SETTLING THE COMPLEXITY OF RECOGNIZING LEAF POWERS

Manuel Lafond, Université de Sherbrooke, Canada

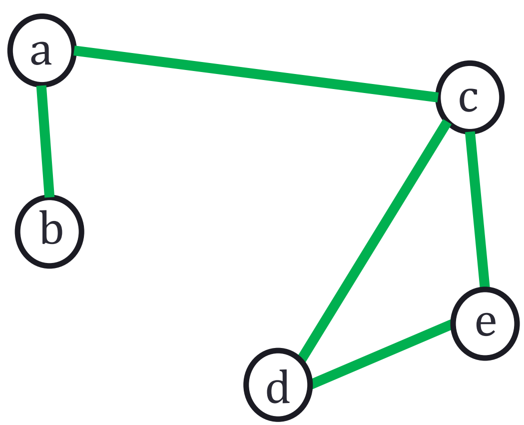
Definition

A graph G is a ***k*-leaf power** if there exists a tree T such that:

- $\boldsymbol{L(T) = V(G)}$, where $L(T)$ is the set of leaves of T
- $\boldsymbol{uv \in E(G) \Leftrightarrow dist_T(u, v) \leq k}$

$3 - leaf\ power\ ?$

G



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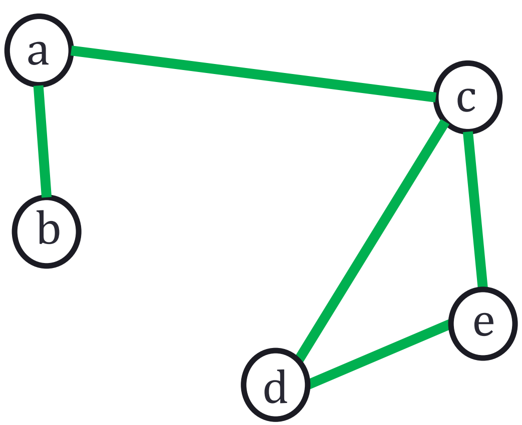
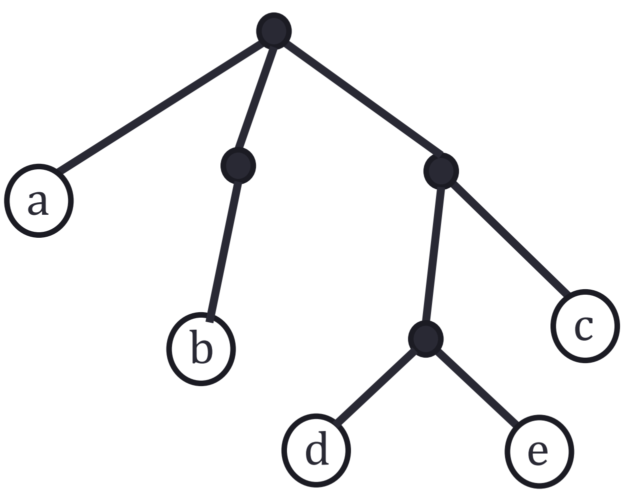


$T:$



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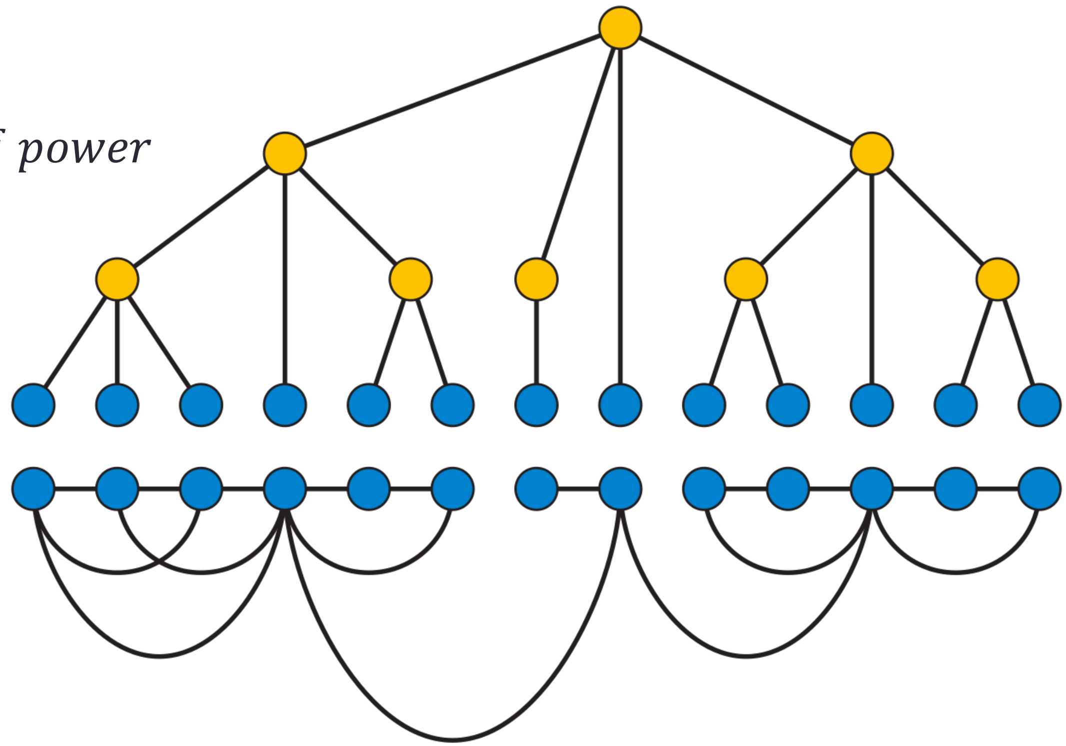
Equivalently, G is a k-leaf power if it can be obtained by taking the k-th power of a tree, and taking the subgraph induced by the leaves of the tree.

$3 - leaf\ power$



Motivations

- Distance matrices are often too noisy.

	G	O	H	M	R
G	0	0.33	0.25	0.52	0.61
O		0	0.31	0.61	0.36
H			0	0.55	0.56
M				0	0.2
R					0

Gibbon (G)

Orangoutan (O)

Human (H)

Mouse (M)

Rat (R)

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- Extract signal from basic info: who is "close", who is "far"?

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	G	O	H	M	R
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H			0	0	0
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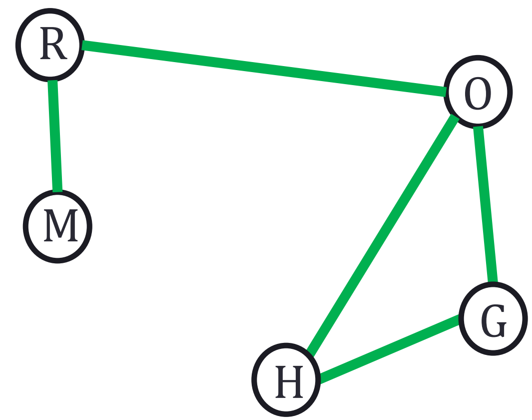

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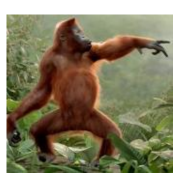

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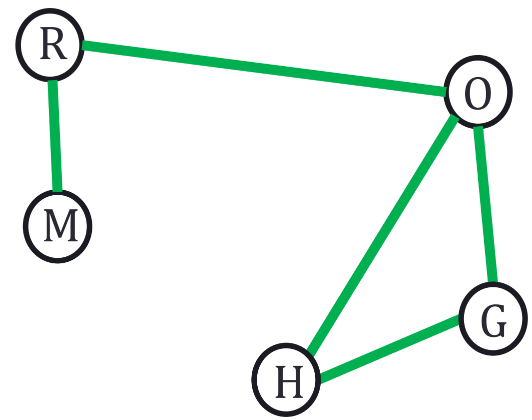
k=2? No

	G	O	H	M	R
G	0	1	1	0	0
O		0	1	0	1
H			0	0	0
M				0	1
R					0

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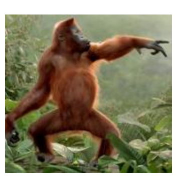
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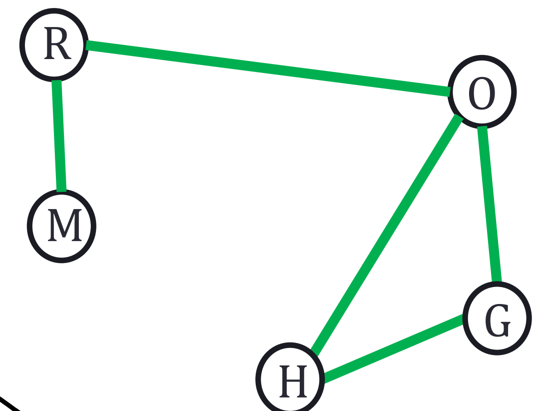
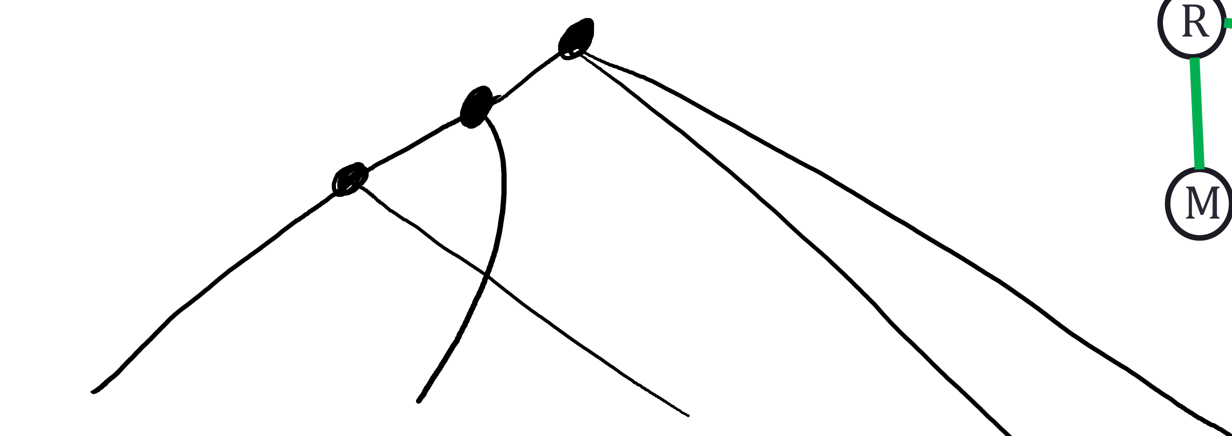
k=3? Yes

	G	O	H	M	R
G	0	1	1	0	0
O		0	1	0	1
H			0	0	0
M				0	1
R					0





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Open problems [Nishimura, Ragde, Thilikos, 2002]

- Can k-leaf powers be characterized by chordal + finite set of forbidden induced subgraphs?
- Complexity of recognizing k-leaf powers if k is in the input?
- Complexity of recognizing k-leaf powers if k is fixed?

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- Can k-leaf powers be characterized by chordal + finite set of forbidden induced subgraphs?
  - YES for $k = 2, 3, 4$ [Brandstatd et al, 2008]. OPEN for $k \geq 5$.
- Complexity of recognizing k-leaf powers if k is in the input?
  - OPEN for 20 years. Not known to be NP-hard or in P.
- Complexity of recognizing k-leaf powers if k is fixed?
  - Poly time for k = 2,3,4,5,6 [Ducoffe, 2019]. OPEN for k > 6.

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- Can k-leaf powers be characterized by chordal + finite set of forbidden induced subgraphs?
  - YES for $k = 2, 3, 4$ [Brandstatd et al, 2008]. OPEN for $k \geq 5$.
  - **ANSWER: NO for all $k \geq 5$. [L, Dupré la Tour, Ndiaye, ICALP 2025]**
- Complexity of recognizing k-leaf powers if k is in the input?
  - OPEN for 20 years. Not known to be NP-hard or in P.
  - **ANSWER: NP-hard. [L, Dupré la Tour, Ndiaye, SODA 2026]**
- Complexity of recognizing k-leaf powers if k is fixed?
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  - **ANSWER: Poly time for all fixed k. [L, SODA 2021]**

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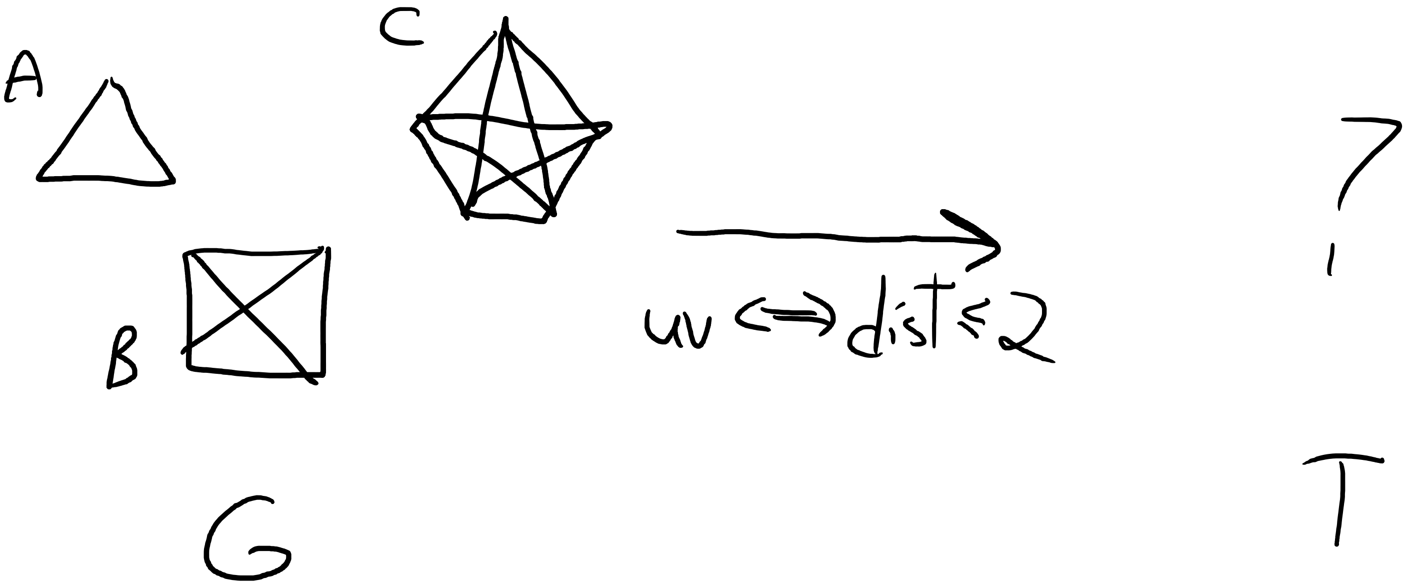
(each CC is a clique)



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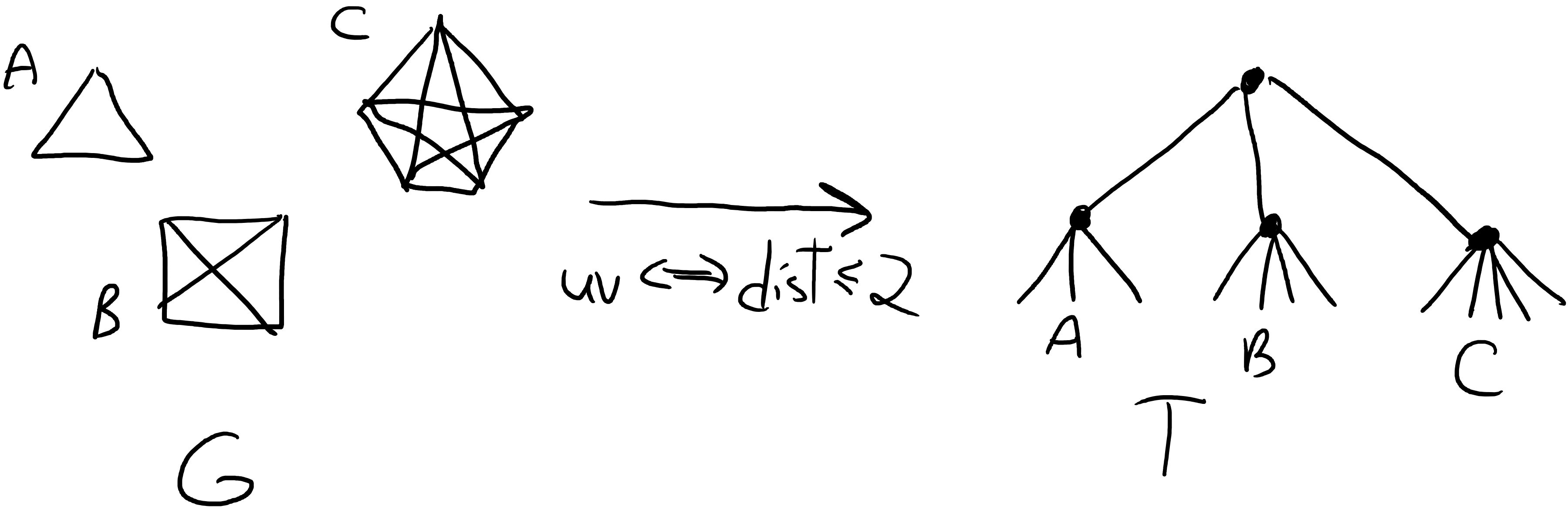

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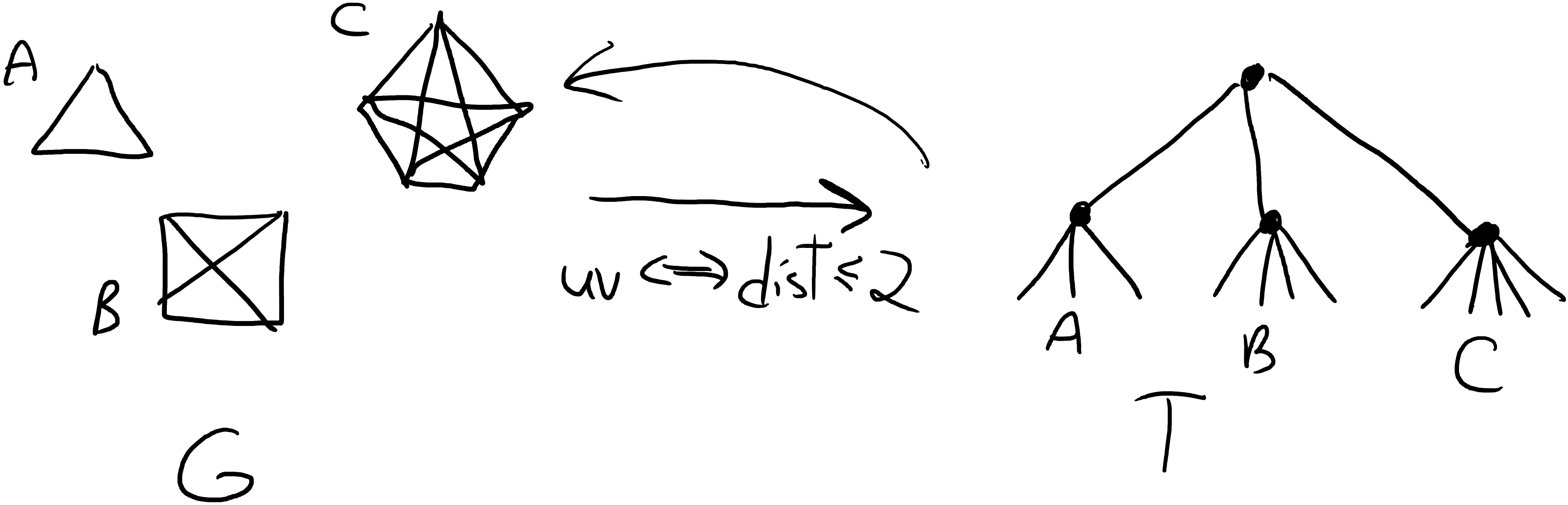

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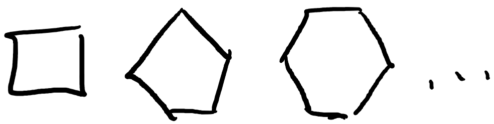
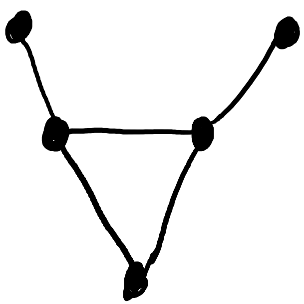
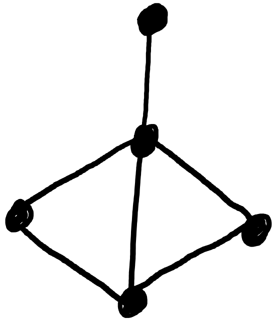
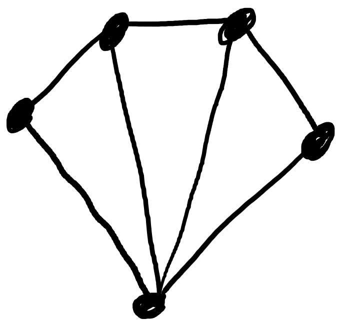
chordal = no 


bull


dart


gem

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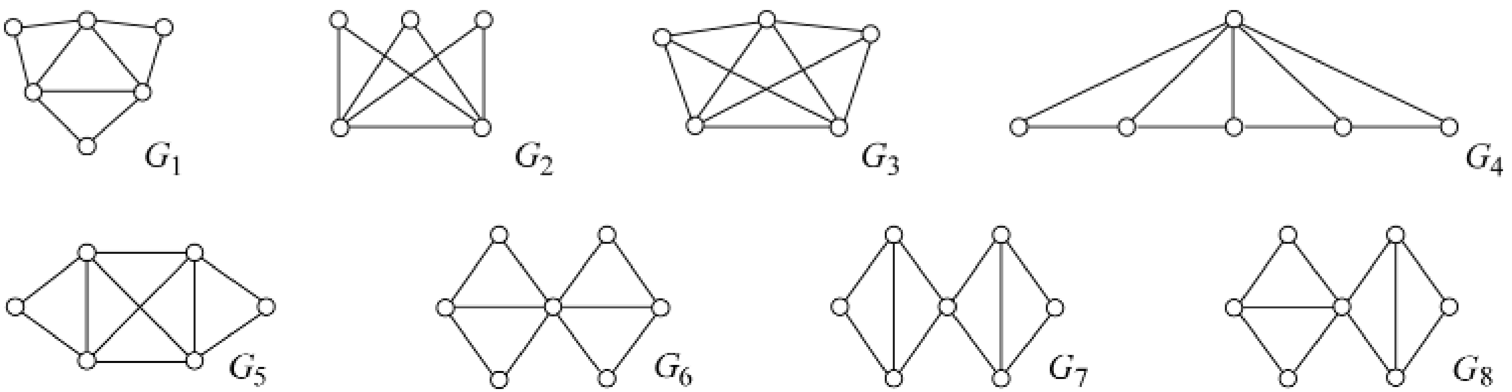

FIG. 1. Forbidden induced subgraphs for chordal basic 4-leaf powers.

1. Can k-leaf powers be characterized by chordal + finite set of forbidden induced subgraphs?
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   - **4-leaf powers** = chordal + X-free, where X is a finite set of forbidden subgraphs [Brandstädt et al., TALG 2008]
   - **Conjecture:** for every integer k, k-leaf powers are exactly the chordal and X_k-free graphs, for some finite set of graphs X_k.

1. Can k-leaf powers be characterized by chordal + finite set of forbidden induced subgraphs?

Document OPEN ACCESS

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k-Leaf Powers Cannot Be Characterized by a Finite Set of Forbidden Induced Subgraphs for k ≥ 5

Authors Max Dupré la Tour, Manuel Lafond, Ndiamé Ndiaye, Adrian Vetta

Part of: Volume: 52nd International Colloquium on Automata, Languages, and Programming (ICALP 2025)

Theorem [ICALP2025]

For $k \geq 5$, the set of k-leaf powers cannot be characterized as the set of strongly chordal graphs which are X_k-free, where X_k is a finite set of graphs

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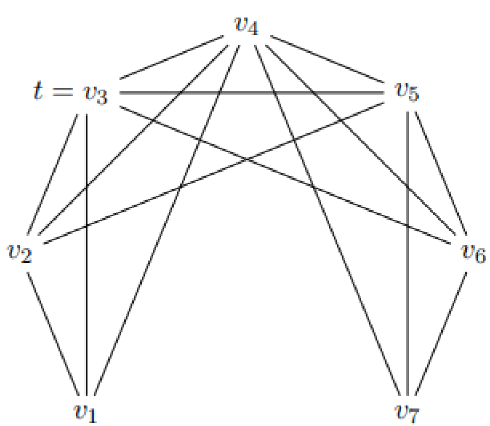
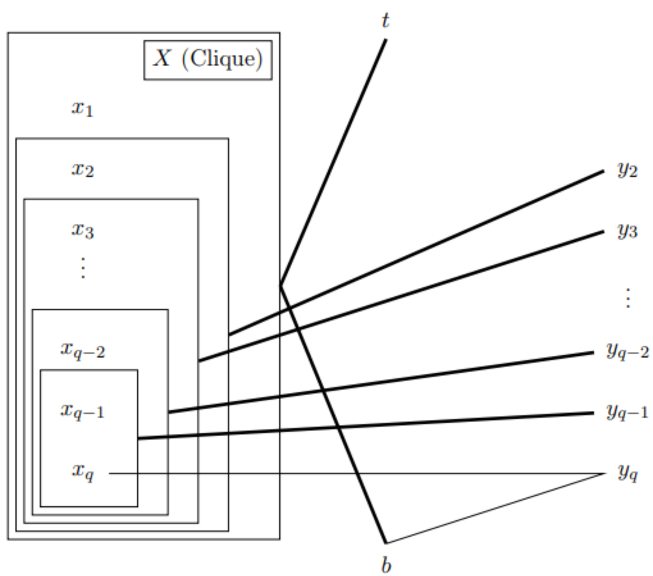
For $k \geq 5$, the set of k-leaf powers cannot be characterized as the set of strongly chordal graphs which are X_k-free, where X_k is a finite set of graphs

Proof technique: construct an infinite set F of strongly chordal graphs that are **minimal non-k-leaf powers**.

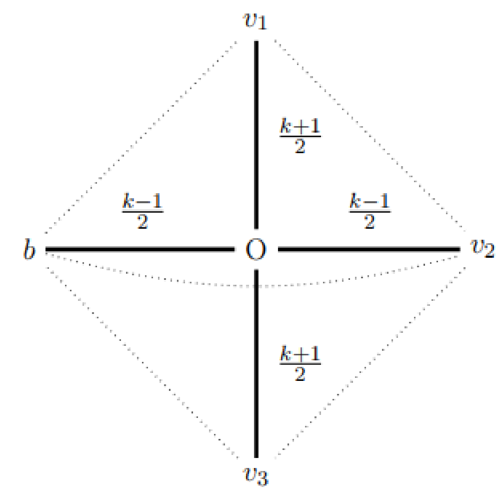
- Meaning, they are not leaf powers, but all their induced subgraphs are.
- Each graph in that set F implies a distinct forbidden induced subgraph, so the finite X_k cannot exist.


v_4
$t = v_3$
v_5
v_2
v_6
v_1
v_7

T

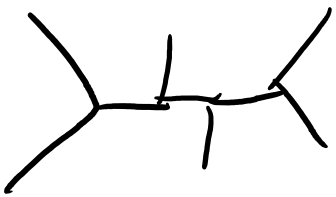

X (Clique)
x_1
x_2
x_3
x_{q-2}
x_{q-1}
x_q
t
y_2
y_3
y_{q-2}
y_{q-1}
y_q
b

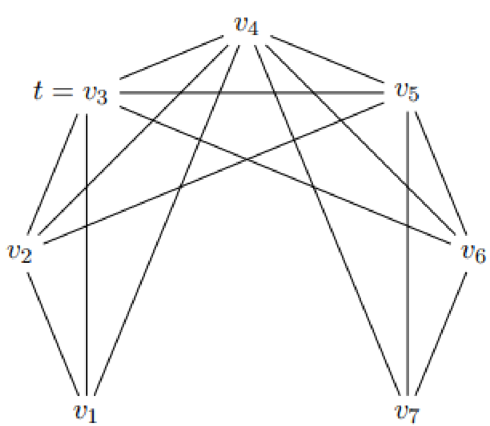
I


v_1
$\frac{k+1}{2}$
$\frac{k-1}{2}$
$\frac{k-1}{2}$
b
O
v_2
$\frac{k+1}{2}$
v_3

B

Unique 5-LP





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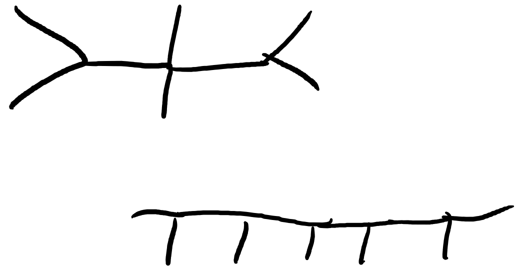
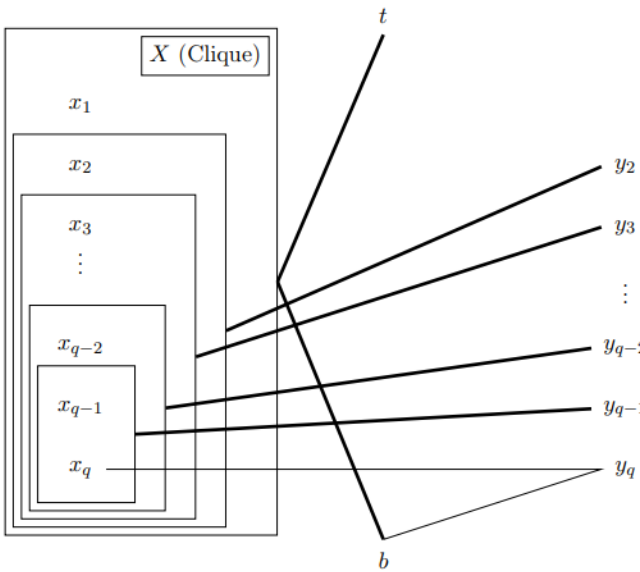
Two 5-LPs





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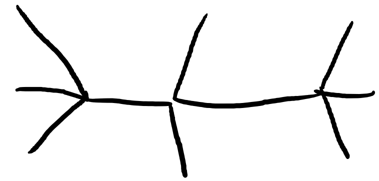

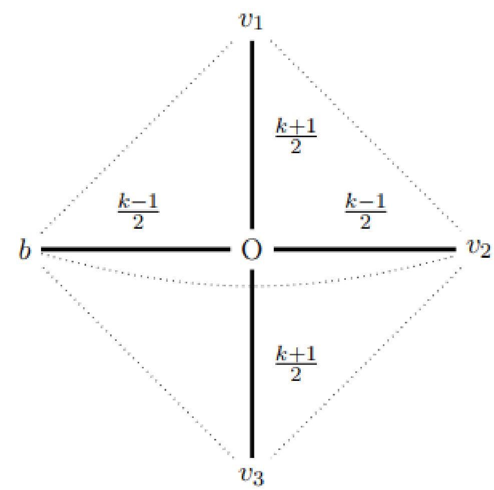

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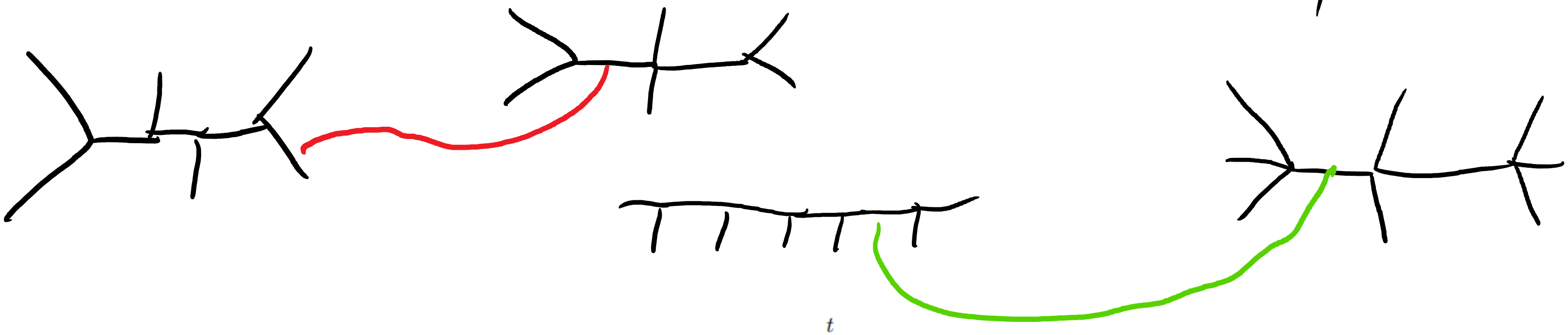
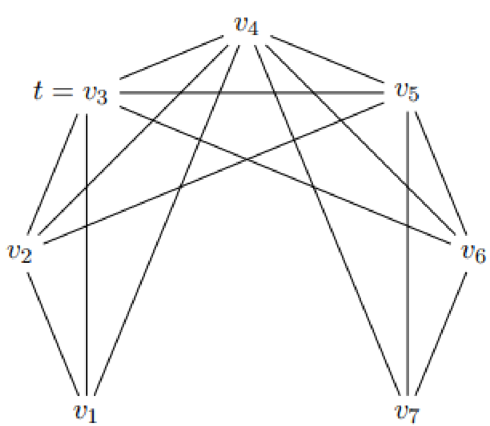
Two 5-LPs

Unique 5-LP





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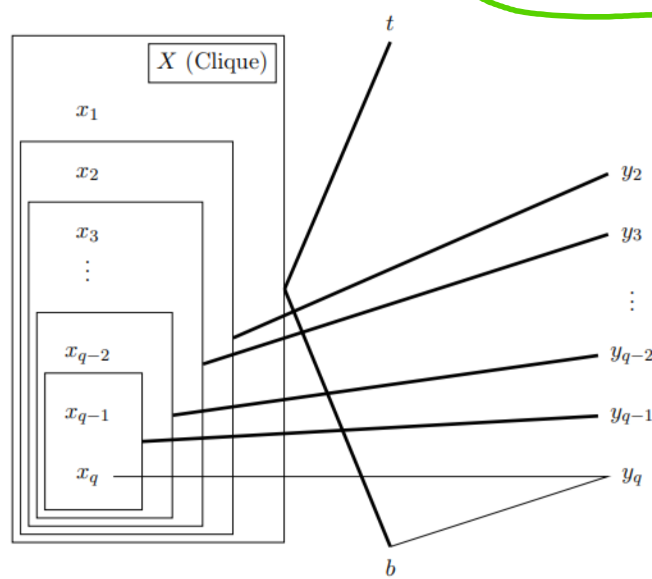

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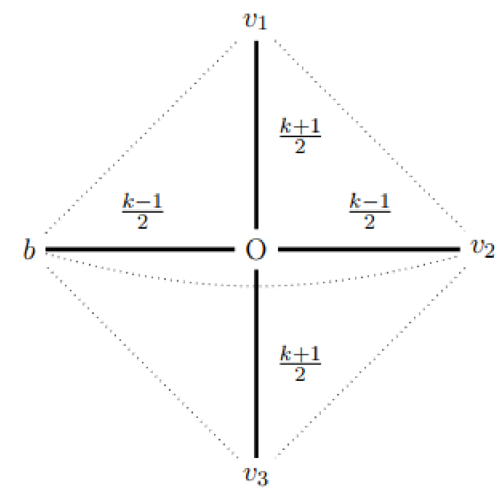

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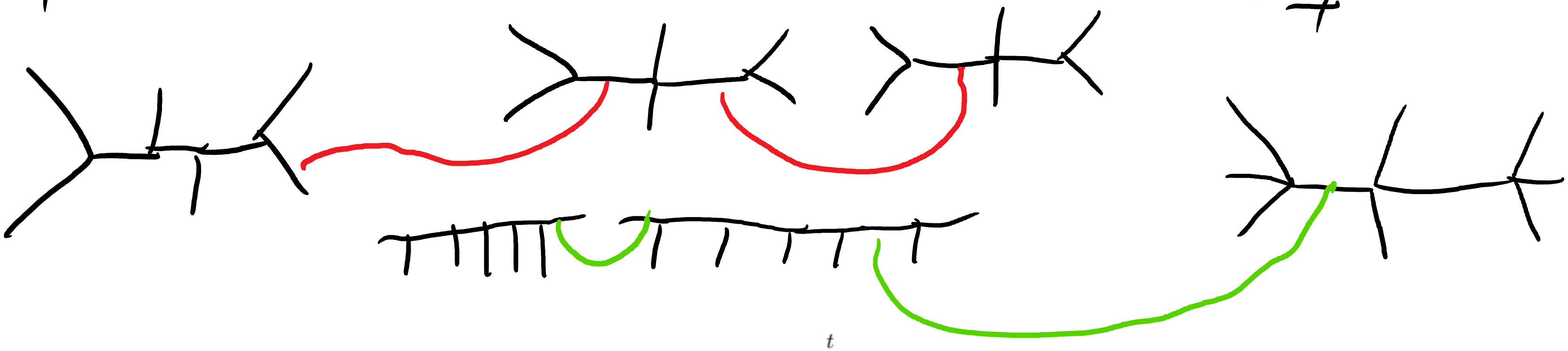
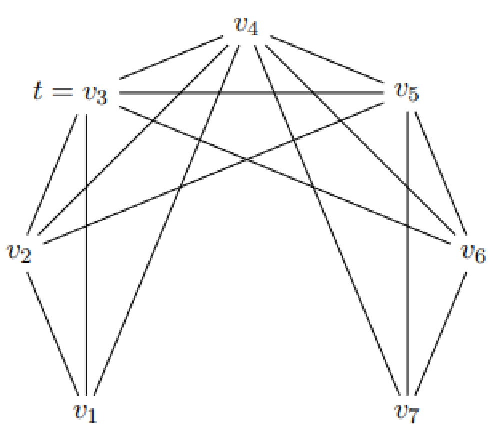
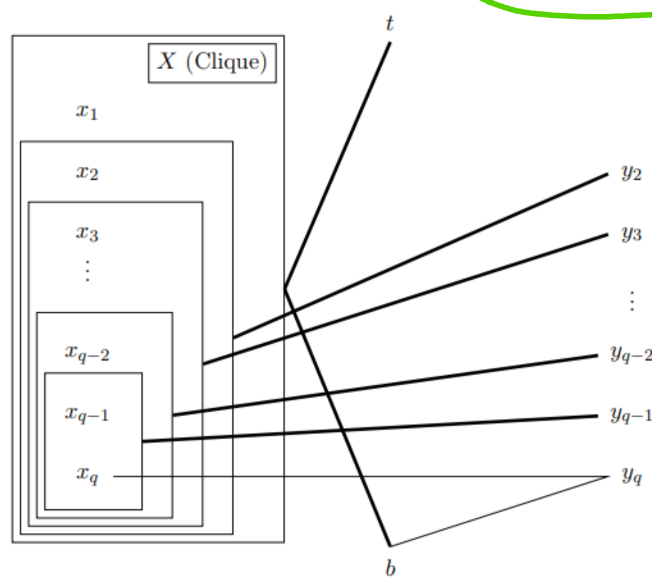
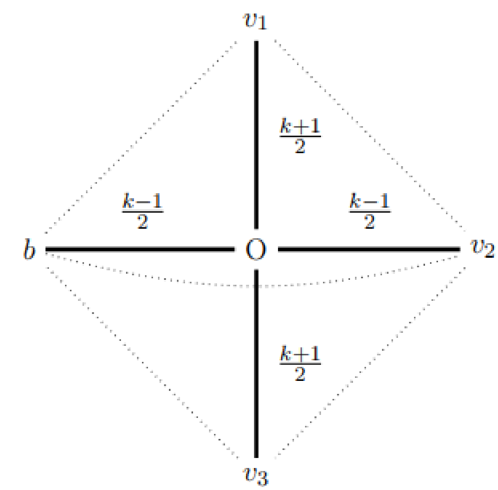
Unique 5-LP Two 5-LPs Unique 5-LP









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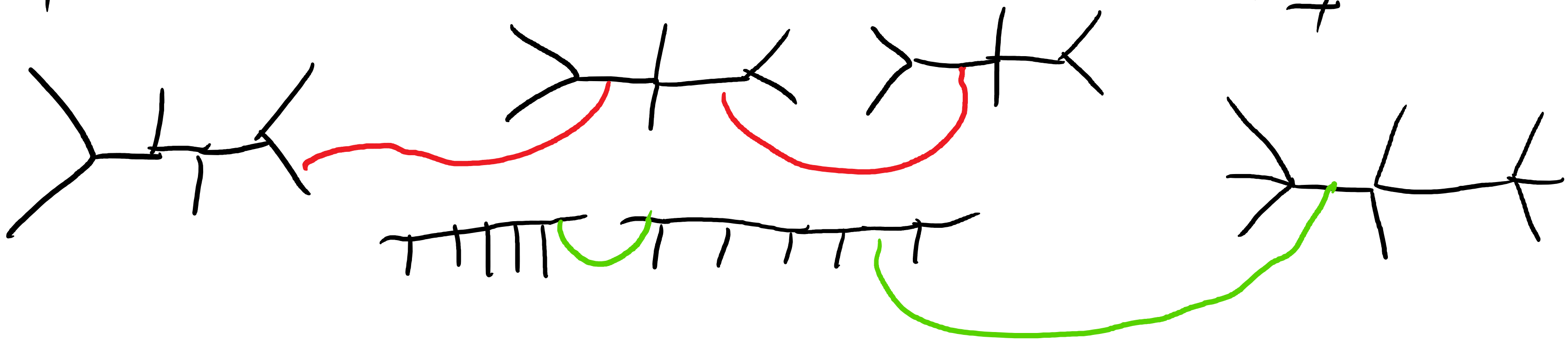
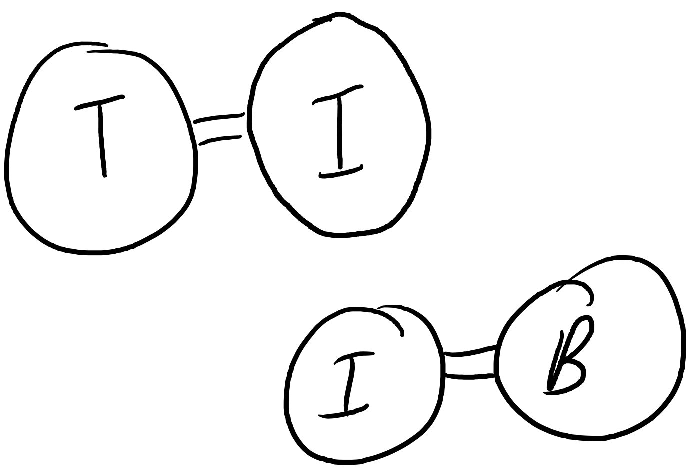
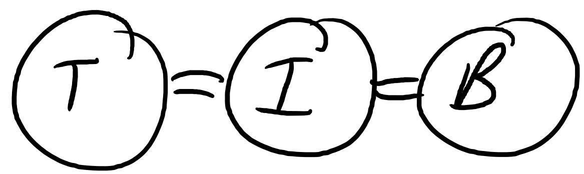
Unique 5-LP   Two 5-LPs   Unique 5-LP



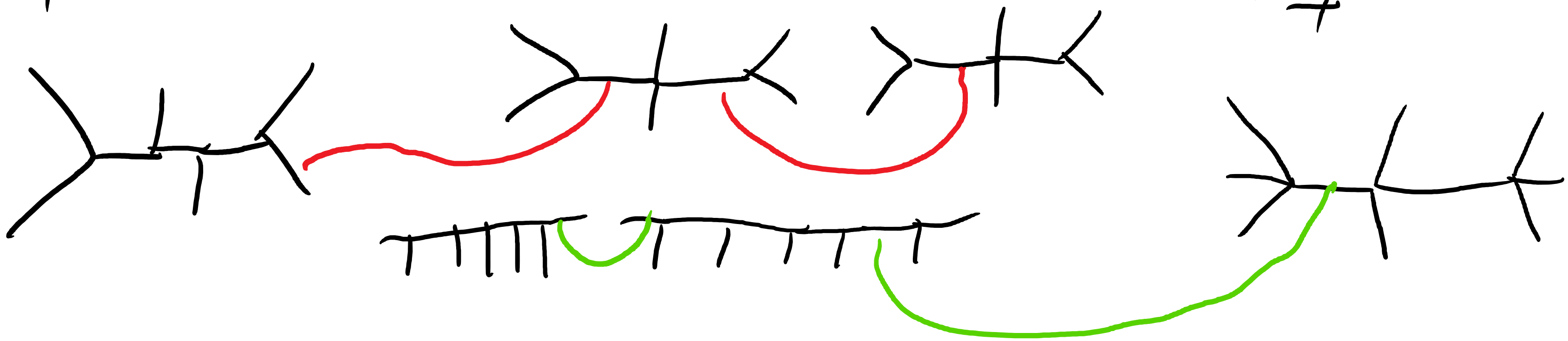
5-LP



Not 5-LP



Unique 5-LP Two 5-LPs Unique 5-LP



5-LP

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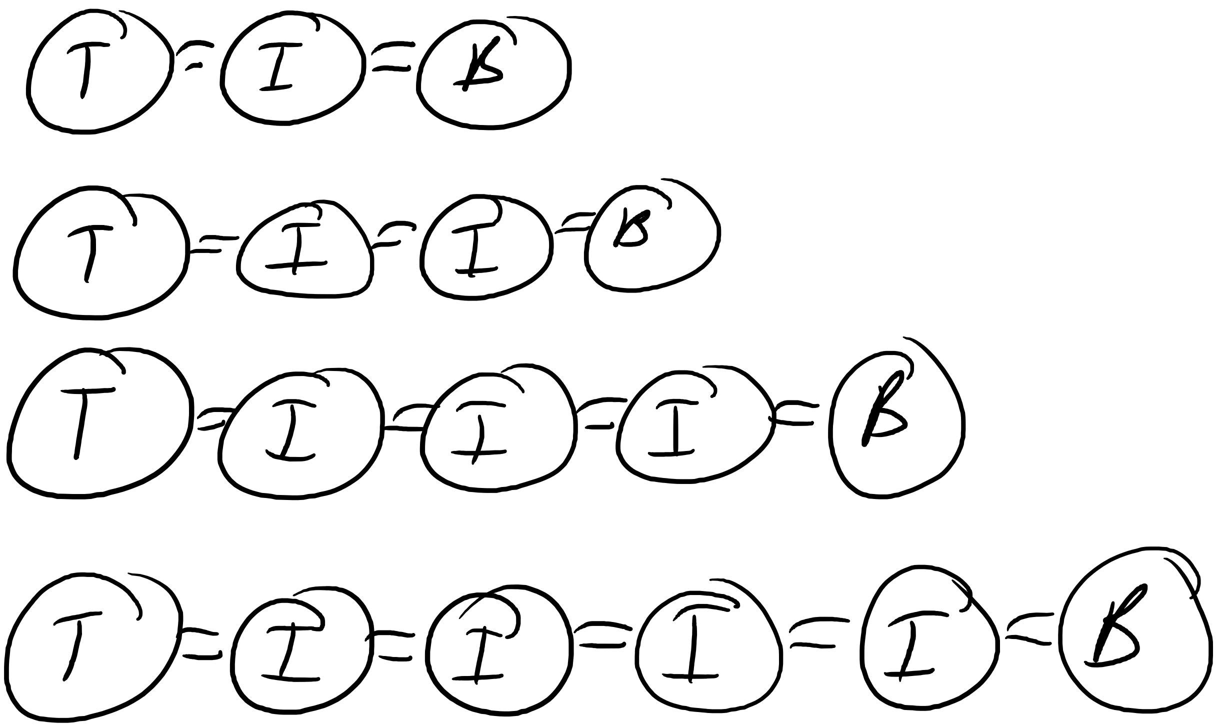
I = I − B

Not 5-LP

T = I = I − B

Infinite list of minimal non-5-LPs



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with some work,
result for k-LP $\Rightarrow$ for $(k+1)$-LP

2. Complexity of recognizing k-leaf powers if k is in the input?

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Recognizing Leaf Powers and Pairwise Compatibility Graphs is NP-Complete

Authors: Max Dupré la Tour, Manuel Lafond, and Ndiamé Ndiaye | AUTHORS INFO & AFFILIATIONS

Theorem [SODA2026]
Given a graph G and an integer k, it is NP-hard to decide whether G is a k-leaf power.

Reduction is from the obscure Triangle Order Clustering problem (TOC) [Kannan, Warnow, 1995], shown NP-hard [Shah, Farach-Colton, 2006].

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TOC problem
Given: ground set S and a partial order $\prec$ on the pairs of S such that, for each $(i, j, k) \in S^3$, $\prec$ a total order on $\{ij, ik, jk\}$.
Goal: find a tree T with leaves $L(T) = S$ such that $ij \prec jk$ implies $dist_T(i,j) < dist_T(j,k)$.

$S = \{1, 2, 3, 4\}$

$12 \prec 13 \prec 23 \quad 14 \prec 34 \prec 13$

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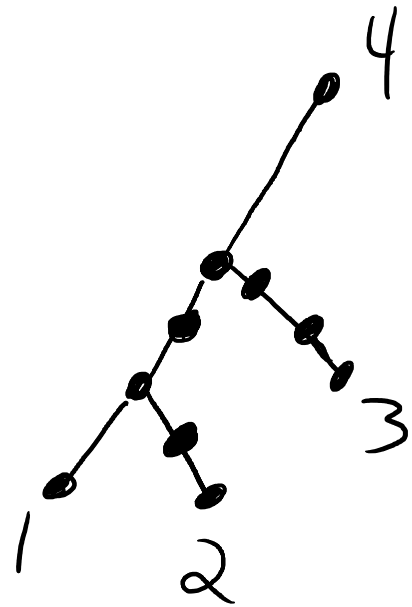

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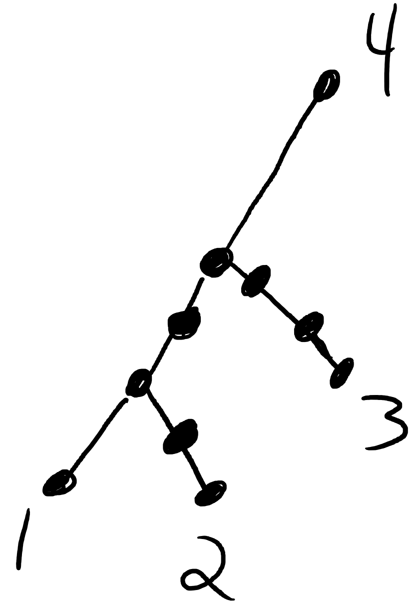
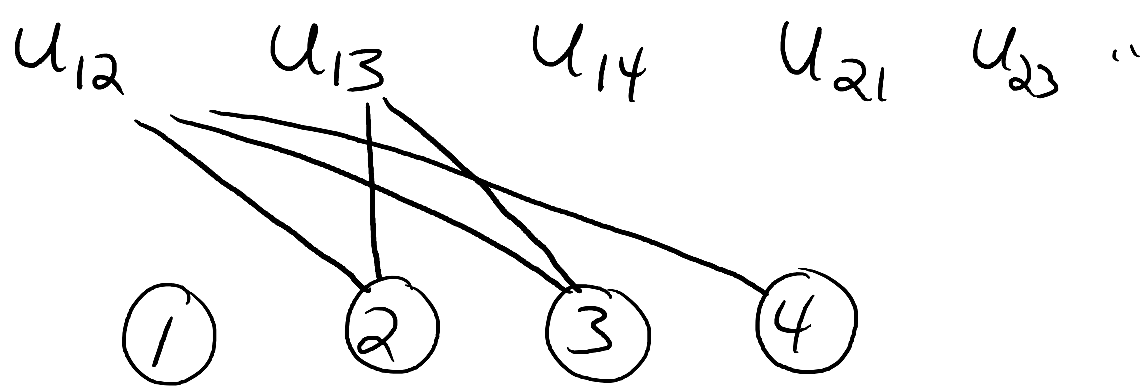


Reduction:



3. Complexity of recognizing k-leaf powers if k is **fixed**.

Theorem [SODA2021]
There is an algorithm that, given a graph G, decides whether G is a k-leaf power in time $O(n^{f(k)})$, where $n = |V(G)|$ and f is a function that depends only on k.

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$$f(k) \simeq \left.k^{k^{k^{k^{\cdot^{\cdot^{\cdot^{k}}}}}}}\right\} k \text{ times}$$

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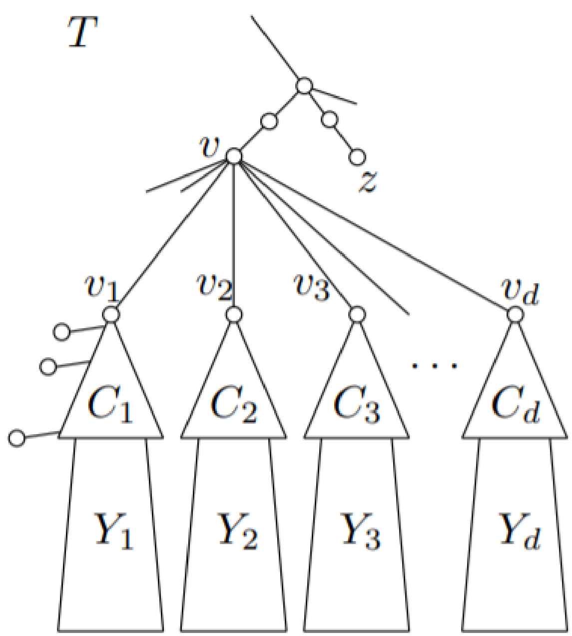

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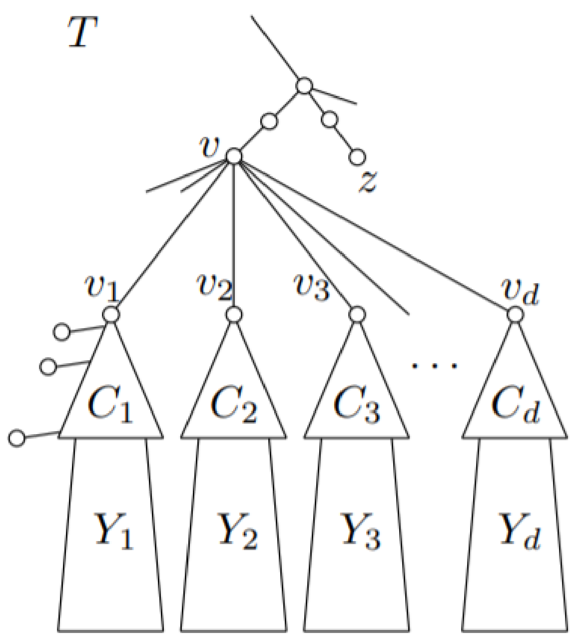

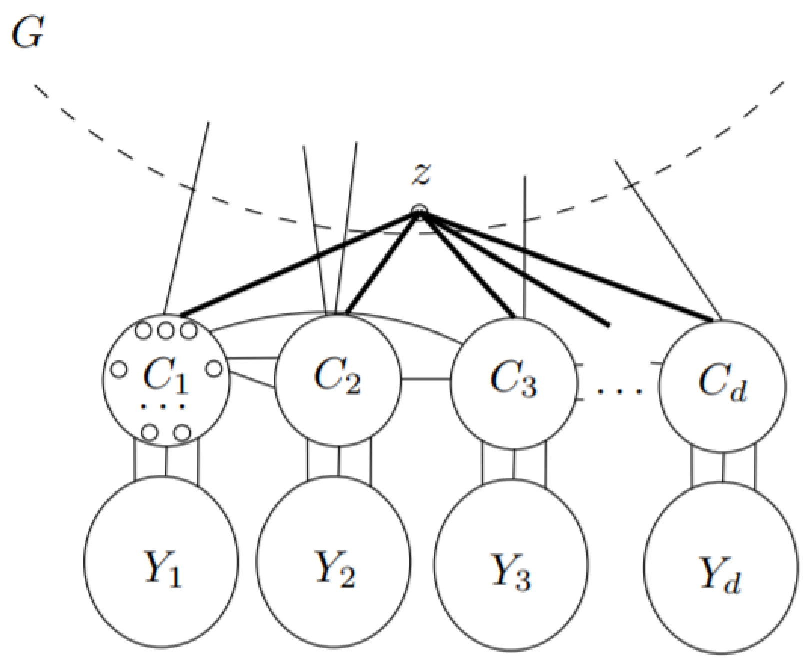

Future work

- Extend Leaf Power definitions to networks.
  - Just one paper: "Structure and Recognition of 3,4-leaf Powers of Galled Phylogenetic Networks in Polynomial Time", Habib and To, 2010
  - Multiple notions of "distance"
  - Are all graphs leaf powers of some network?
  - Restriction to network classes.
- Allow negative edge weights in trees
  - Complexity? Characterization? Especially if we require caterpillars.
  - G admits a negative leaf power that is a caterpillar if and only if G has linear-mim-width 1 (proof in an email somewhere).

Future work

- G is a Pairwise Compatibility Graph (PCG) if there exist a tree T with leaves $V(G)$ and thresholds a, b such that $uv \in E(G) \Leftrightarrow dist_T(u, v) \in [a, b]$.
  - Complexity for fixed a, b? (non-fixed = NP-hard)
  - Characterize PCGs of caterpillars.
  - Many other open problems.