# ON THE CHARACTERIZATION OF PAIRWISE COMPARABILITY GRAPHS

Manuel Lafond, Université de Sherbrooke, Canada

UNIVERSITÉ DE SHERBROOKE

**Definition**

A graph $G$ is a $\boldsymbol{Pairwise\ Comparability\ Graph}$ $(\boldsymbol{PCG})$ if there exist a tree $T$ and non-negative integers $d_1, d_2$ such that:

- $Leaves(T) = V(G)$, where $Leaves(T)$ is the set of leaves of $T$
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$d_T(u, v) =$ distance between $u$ and $v$ in $T$.

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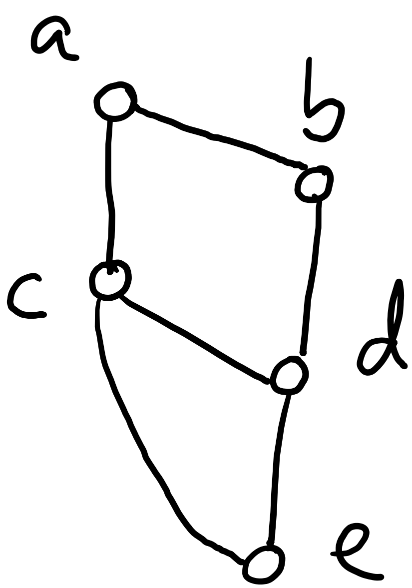

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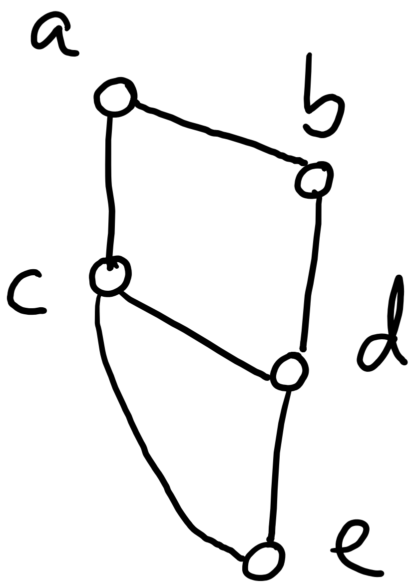


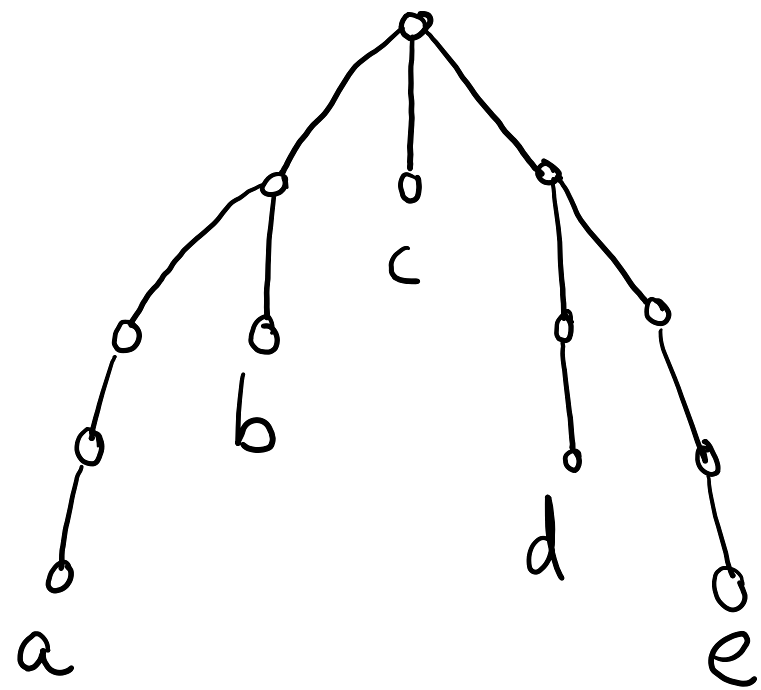


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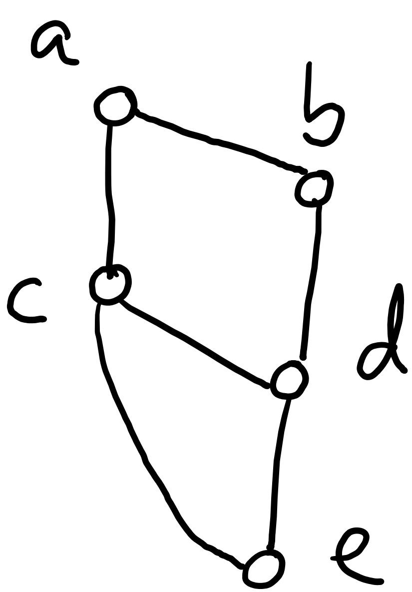


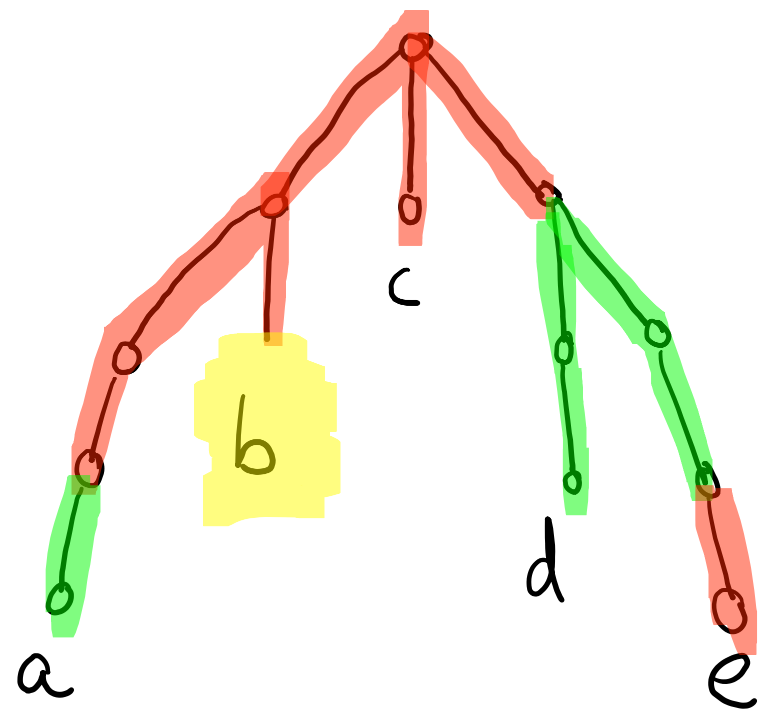


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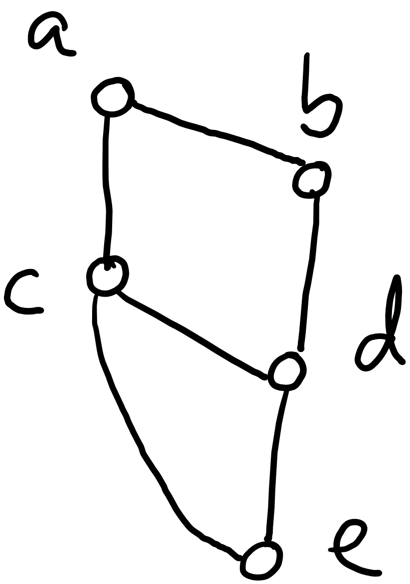


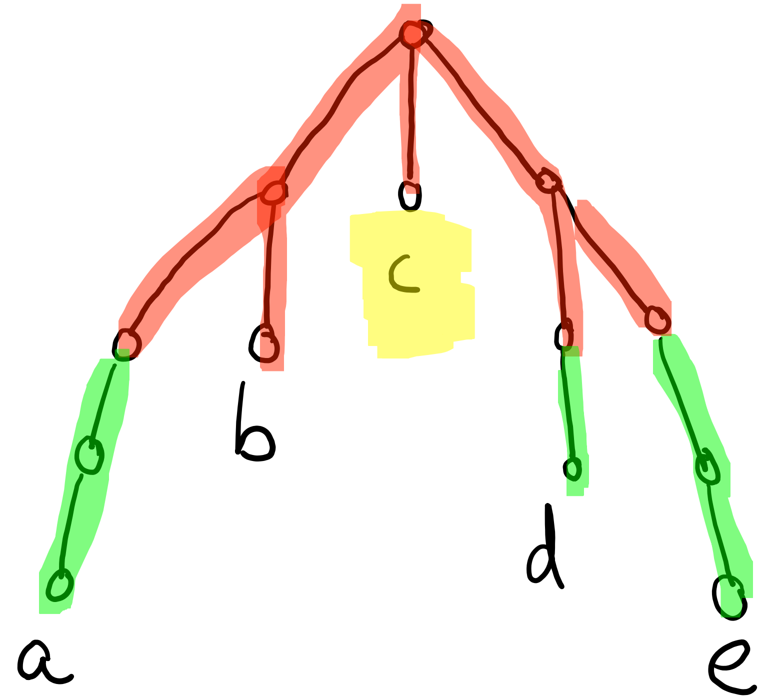


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Such a tree $T$, if it exists, is called a $[d_1, d_2]$-PCG tree for $G$.

We say that $G$ is a $[\boldsymbol{d_1}, \boldsymbol{d_2}]$**-PCG** if there exists a $[d_1, d_2]$-PCG tree for $G$.

PCG $= \bigcup_{d_1,d_2} [d_1, d_2]$-PCG

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**Open problem 1**: characterize PCGs.

**Open problem 2**: determine the complexity of deciding whether a given graph $G$ is a PCG?

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Introduced in [Kearney et al, 2003]

- Motivation: reconstruct phylogenies from species distances
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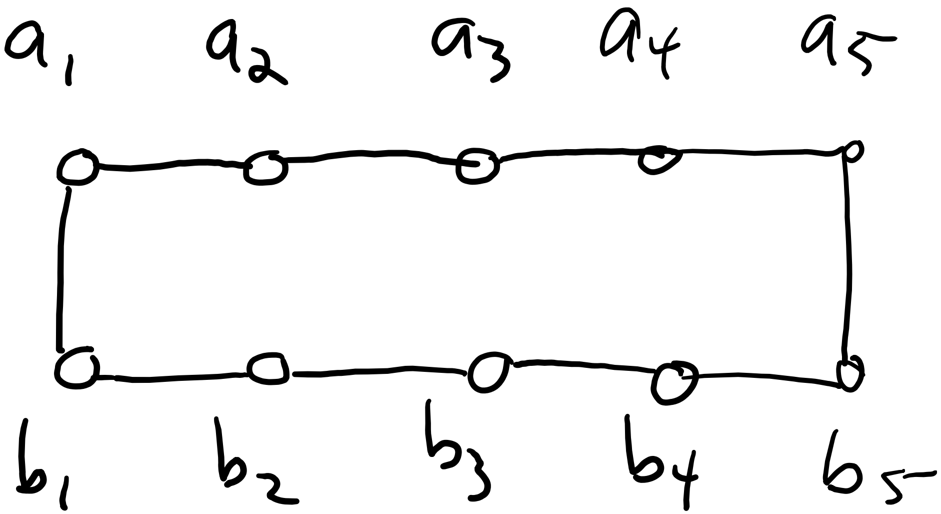
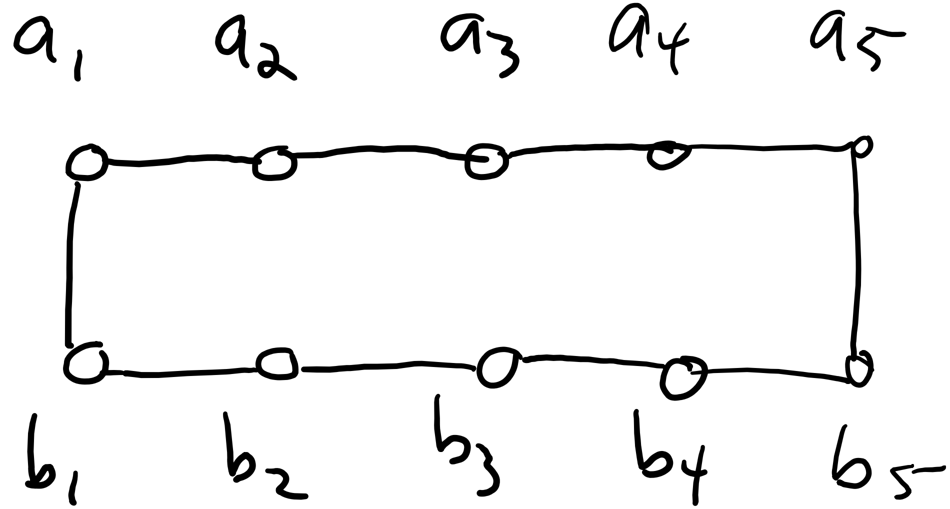



[5,7]-PCG

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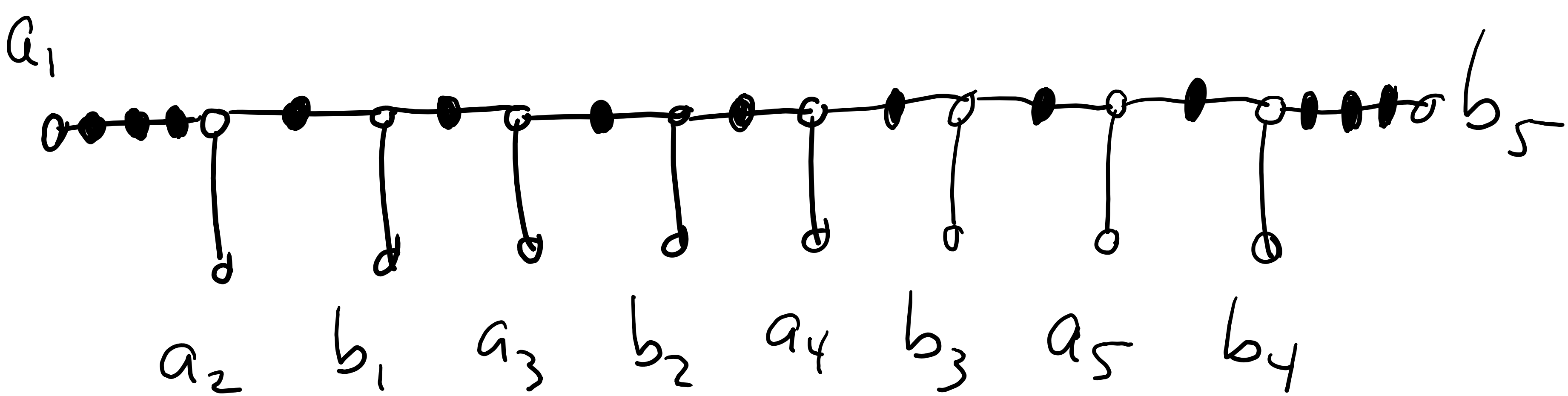
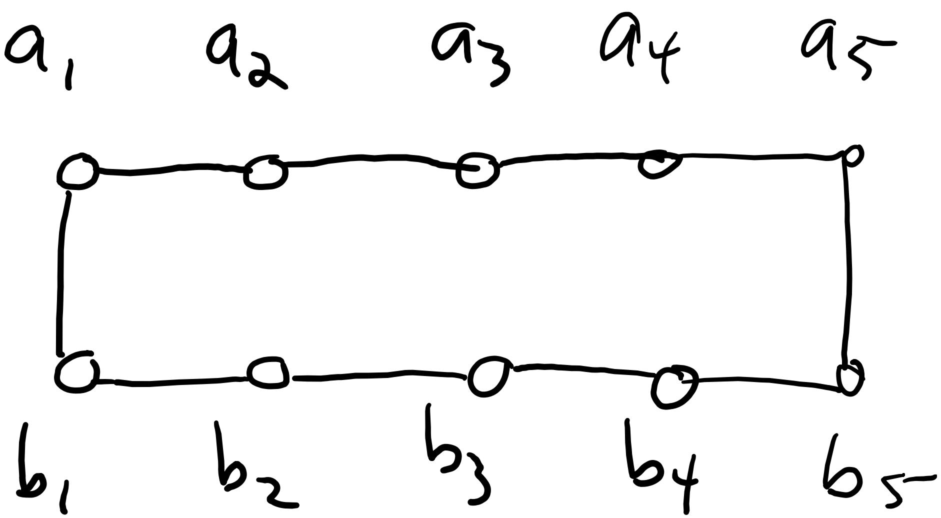
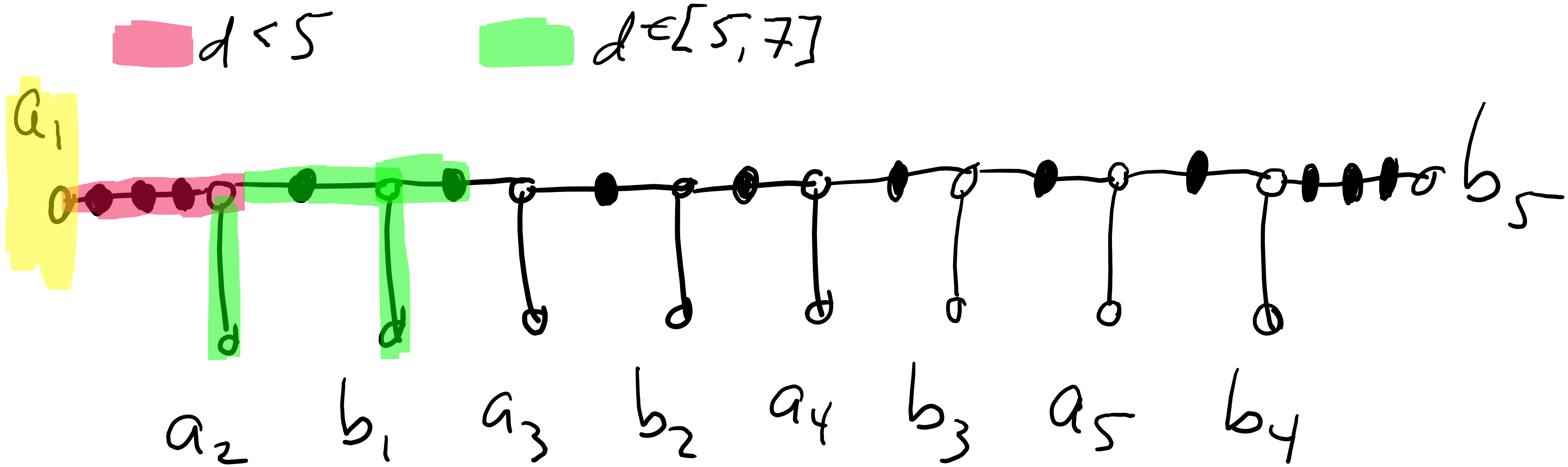
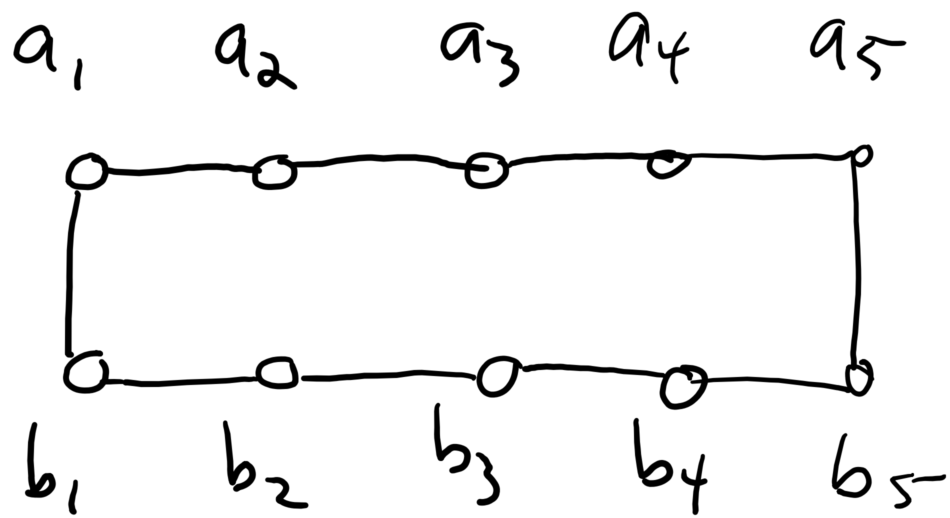


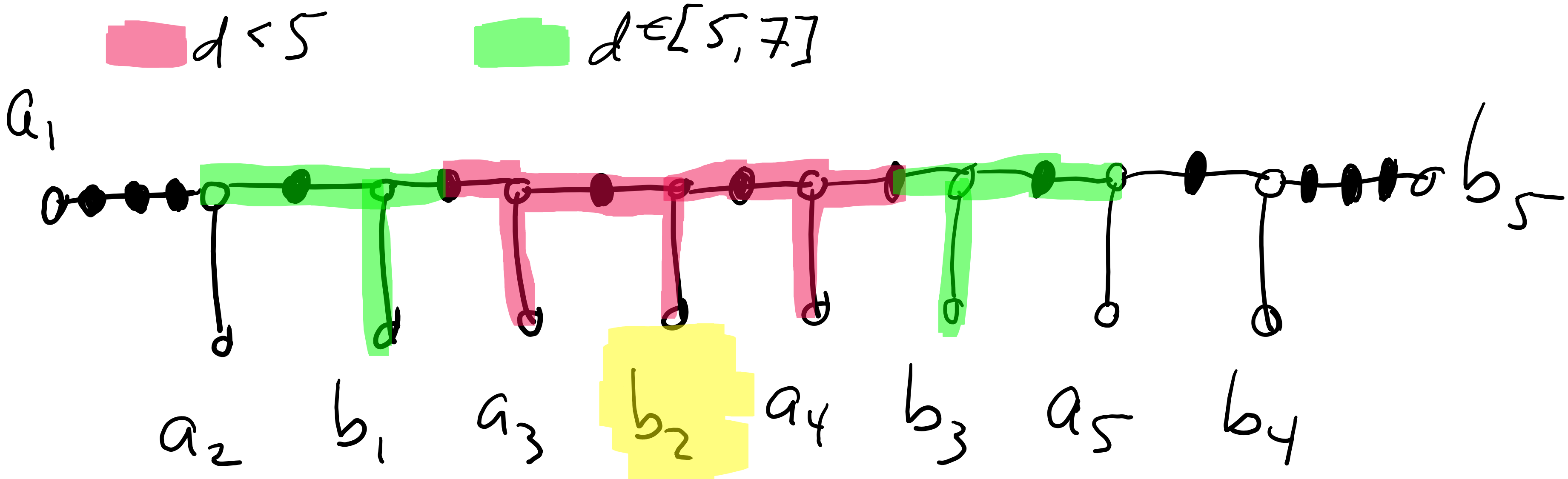



$[5,7]$-PCG






# [5,7]-PCG



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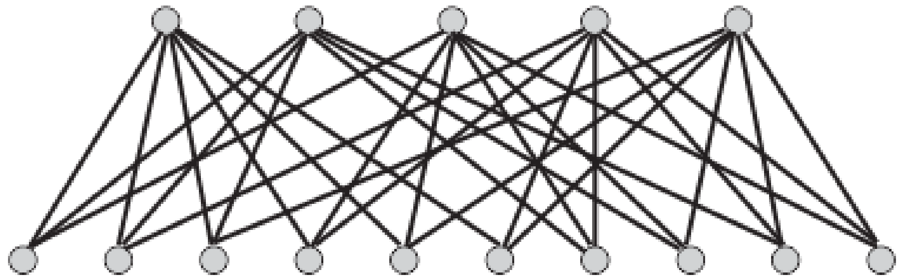
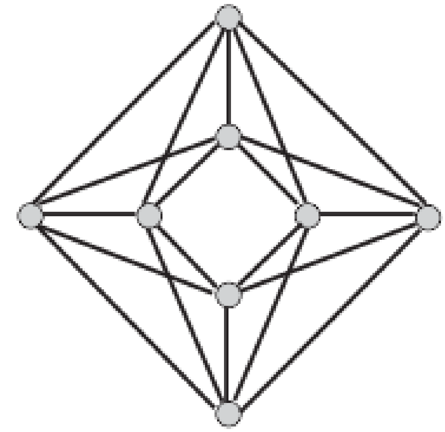
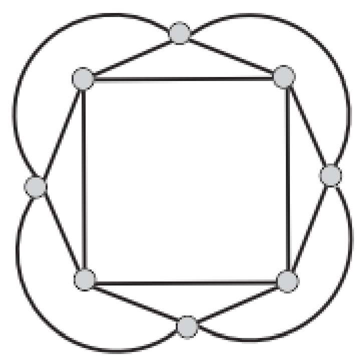
Not all graphs are PCGs [Yanhaona et al, 2010]
- There exists a graph on 8 vertices that is not a PCG [Durocher et al., 2015]
- Not all planar graphs are PCGs
- Not all bipartite graphs are PCGs



(a)



(b)



(c)

**Theorem** [Hossain et al, 2016]
Suppose that $G^c$ contains two fully disjoint induced cycles (i.e. no edge has its endpoints on both cycles) of length at least 4. Then $G$ is not a PCG.

$G^c$ = complement of $G$.

**Theorem** (lesser version)
Suppose that $G^c$ contains an induced $2C_4$. Then $G$ is not a PCG.

$G^c$ $=$complement of $G$.
$2C_4$ **=** two vertex-disjoint induced cycles of length 4, without edges having endpoints in both cycles.

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$G$:

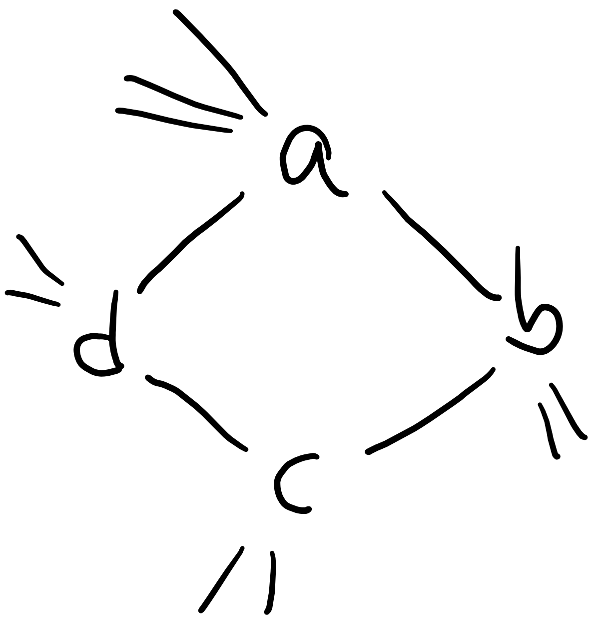
$G^c$

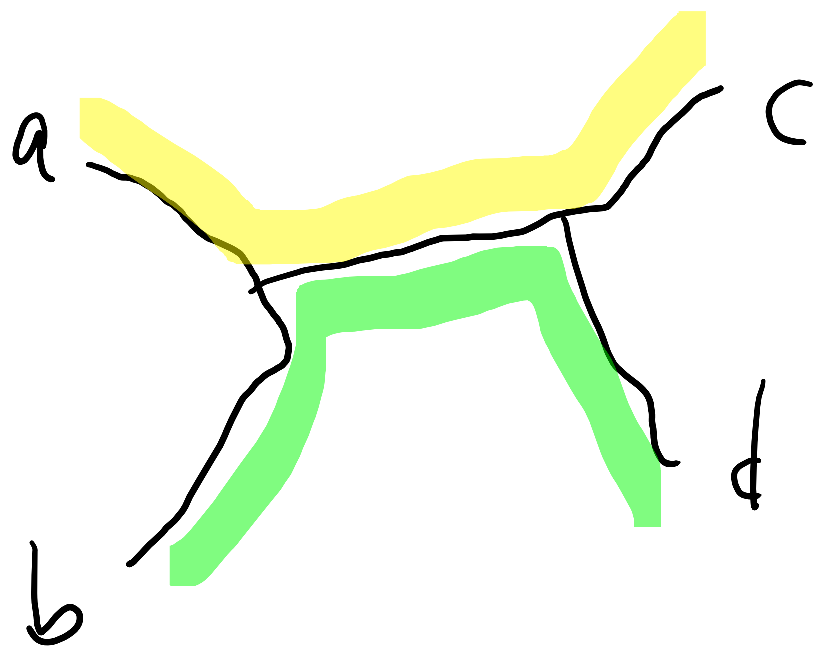
$2C_4$

**Lemma 1**

Suppose that $G$ contains an induced $C_4$ with edges $ab, bc, cd, da$ and that $G$ admits a $[d_1, d_2]$-PCG-tree $T$. Then $d_T(a, c) < d_1$ or $d_T(b, d) < d_1$.

(or both)



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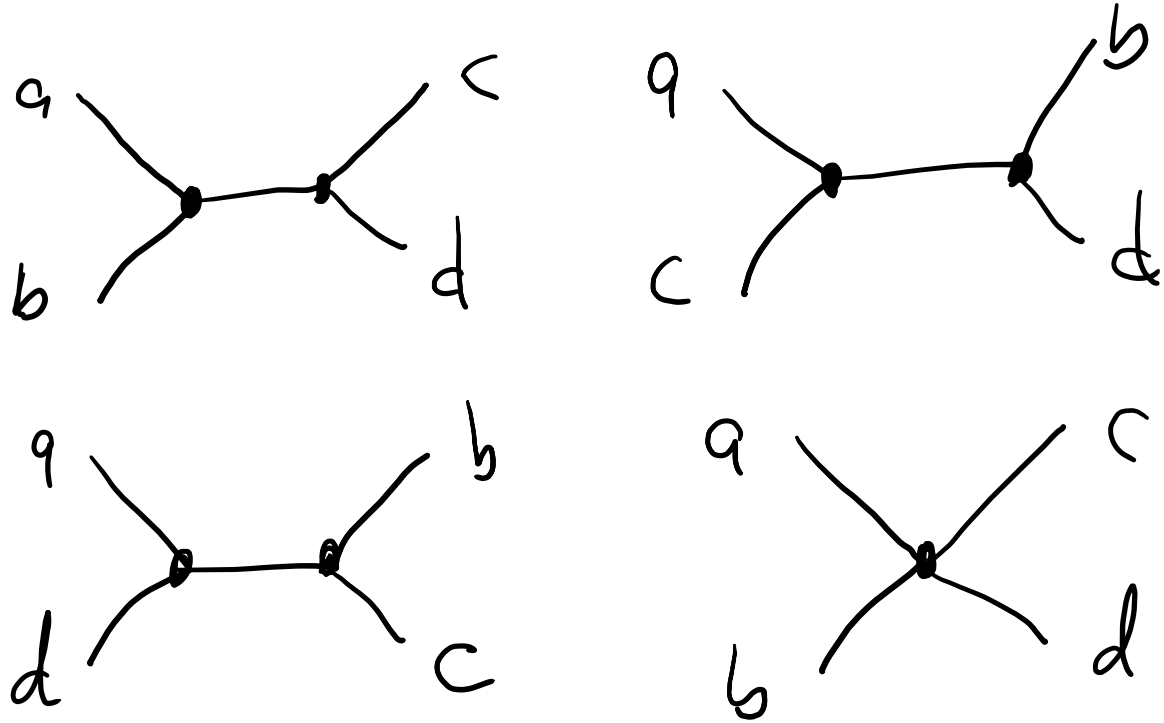
*Proof.* Suppose $d_T(a,c) \geq d_1$ and $d_T(b,d) \geq d_1$. Because $ac \notin E(G)$, we have $d_T(a,c) > d_2$ and likewise $d_T(b,d) > d_2$.
There are four possible topologies for $a, b, c, d$ in $T$.



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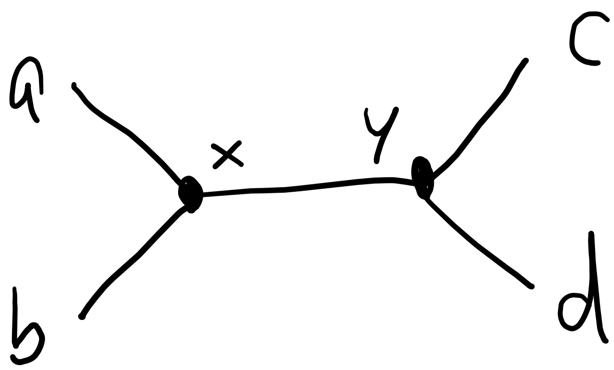

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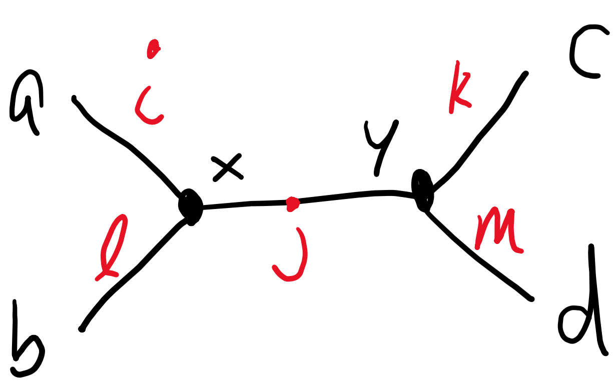


$i = d_T(a,x)$

$j = d_T(x,y)$

$k = d_T(y,c)$

$l = d_T(b,x)$

$m = d_T(y,m)$

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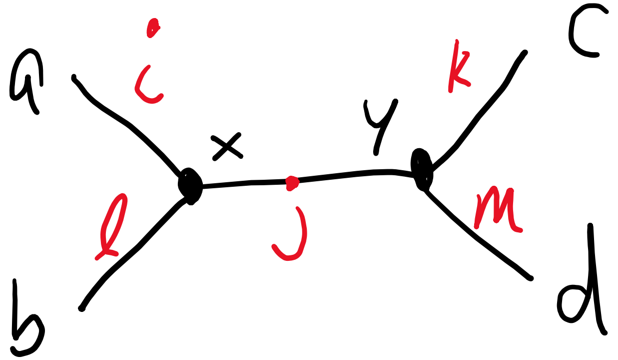



$$d_T(a,c) + d_T(b,d)$$
$$= i + j + k + l + j + m > 2d_2$$

Since $ad, bc$ are edges of the $C_4$,
$d_T(a,d) \leq d_2$ and $d_T(b,c) \leq d_2$, so

$$d_T(a,d) + d_T(b,c)$$
$$= i + j + m + l + j + k \leq 2d_2$$

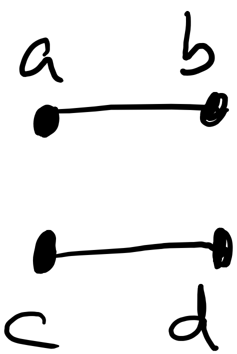
Contradiction.
Other topologies = similar.

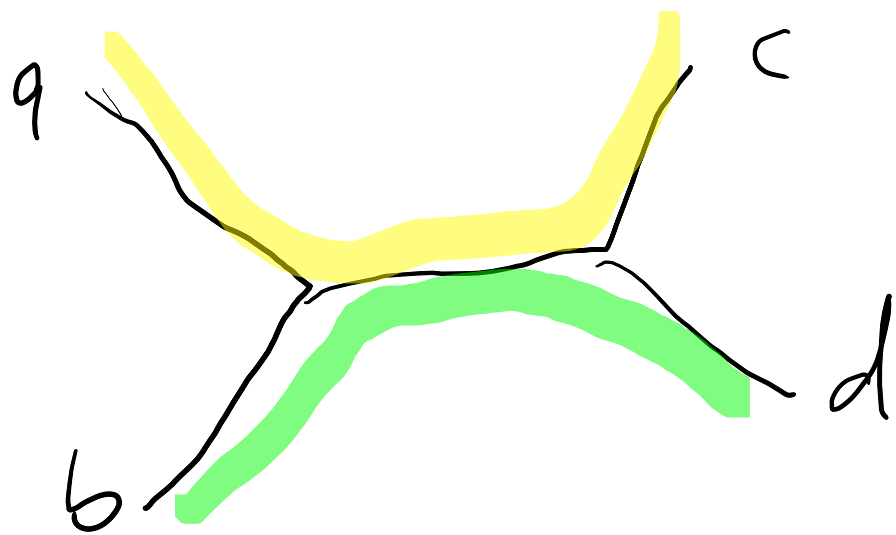
**Lemma 1**

Suppose that $G$ contains an induced $C_4$ with edges $ab, bc, cd, da$ and that $G$ admits a $[d_1, d_2]$-PCG-tree $T$. Then $d_T(a, c) < d_1$ or $d_T(b, d) < d_1$.

**Lemma 2**

Suppose that $G$ contains an induced $2K_2$ with edges $ab, cd$ and that $G$ admits a $[d_1, d_2]$-PCG-tree $T$. Then at least one of $d_T(a, c), d_T(a, d), d_T(b, c), d_T(b, d)$ is $> d_2$.

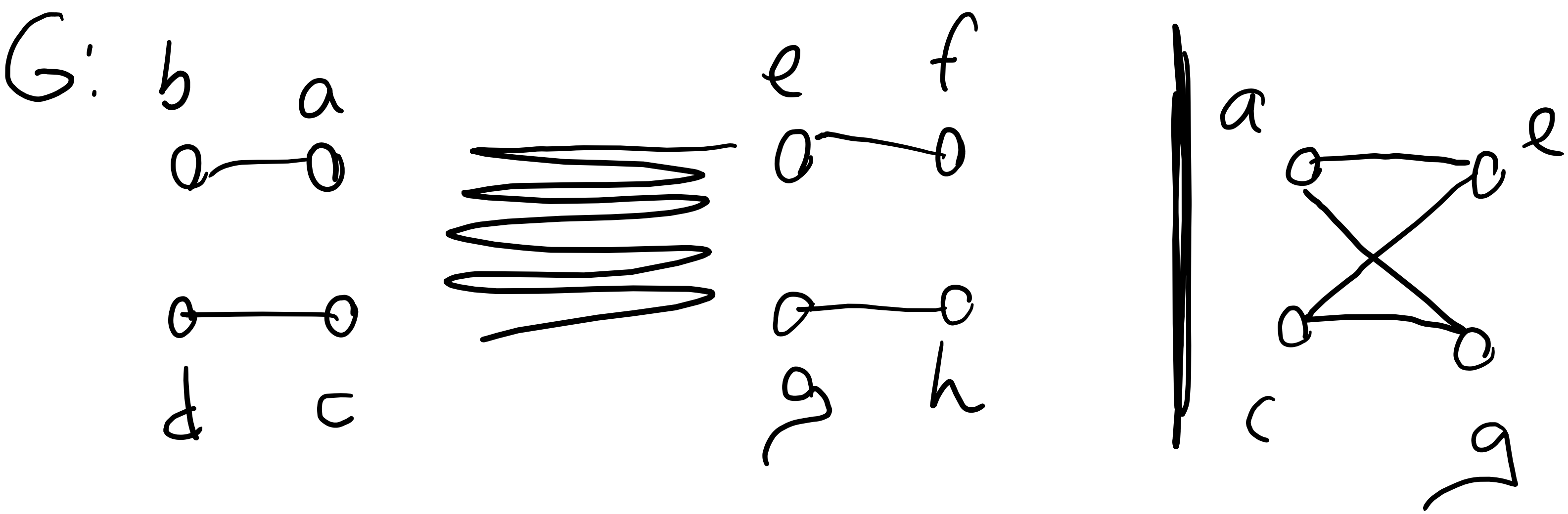


**Theorem** (lesser version)
Suppose that $G^c$ contains an induced $2C_4$. Then $G$ is not a PCG.

*Proof of the theorem.*

- Suppose there is a $[d_1, d_2]$-PCG tree $T$ for $G$.
- Since $G^c$ has a $2C_4$, $G$ has two $2K_2$'s, with all edges between them.
- By Lemma 2, each $2K_2$ has a non-edge, say $ac$ and $eg$, such that $d_T(a,c) > d_2$ and $d_T(e,g) > d_2$.
- But $a, c, e, g$ induce a $C_4$ and by Lemma 1, one of these distances must be $< d_1$.



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**Theorem** [Hossain et al., 2016]
Suppose that $G^c$ has no cycle. Then $G$ **is** a PCG.

Many open problems summarized in two surveys of
[Calamoneri & Sinaimeri, 2016] and [Rahman & Ahmed, 2020]

**Open problem**: characterize $[d, d]$-PCG graphs, where $d$ is a fixed integer, or characterize $\cup_d\ [d, d]$-PCG.

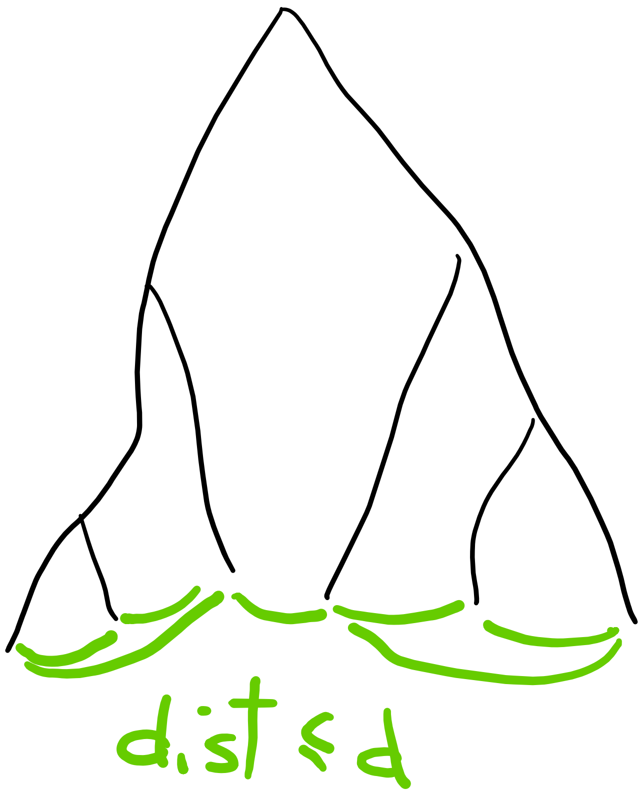
- Known for d $=$ 2,3,4 [Brandstädt et al, 2010]

**Open problem**: for $G$ a PCG, what is the minimum $d_2$ such that $G$ is a $[d_1, d_2]$-PCG? How large must $d_2$ be in the worst case.

- Belief: some PCGs require $d_2 \in \Omega(\sqrt{n})$.
- Weak belief: the worst case is $\Theta(n)$.
- Superexponential $d_2$ has not been excluded.

# What if $d_1 = 0$?

- The set of $[0, d]$-PCGs, for all $d$, is equivalent to a graph class called **leaf powers** (graphs induced by the leaves of a $d$-th power of a tree).

# What if $d_1 = 0$?

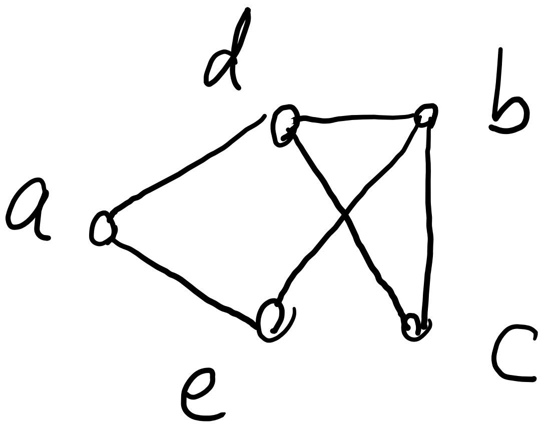
- $[0, 2]$-PCGs = cluster graphs (each CC is a clique)
- $[0, 3]$-PCGs = chordal + (bull, dart, gem)-free graphs [Brandstädt et al, 2008]
- $[0, 4]$-PCGs = chordal + X-free, where X is a finite set of forbidden induced subgraphs [Rautenbach, 2006] (|X| = unspecified)
- $[0, 5]$-PCGs = ???

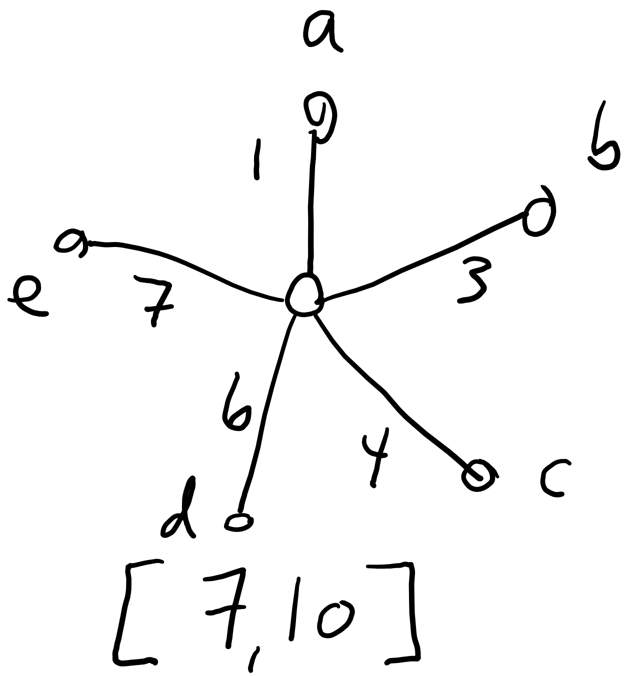
- **Open problem**: is it true that for any fixed $d$, $[0, d]$-PCGs can be characterized as chordal + X-free graphs, with X finite?

- For any fixed $d$, $[0, d]$-PCGs can be recognized in polynomial time [L, 2022]

# What if we restrict $T$?

- $G$ is a *star-PCG* if it admits a $[d_1, d_2]$-PCG tree that is a subdivision of a star for some $d_1, d_2$.



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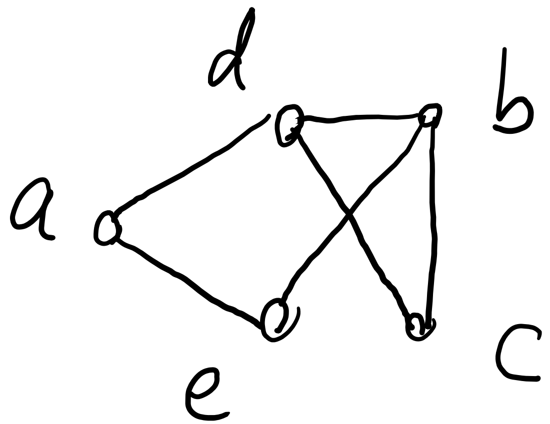


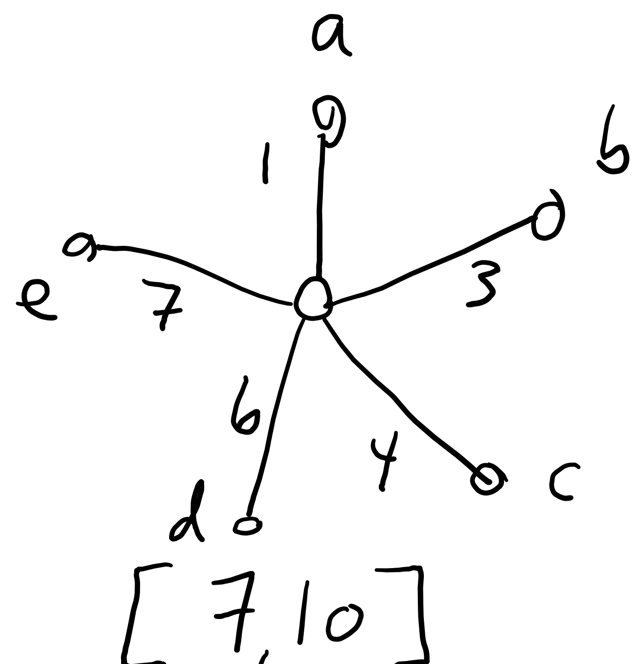


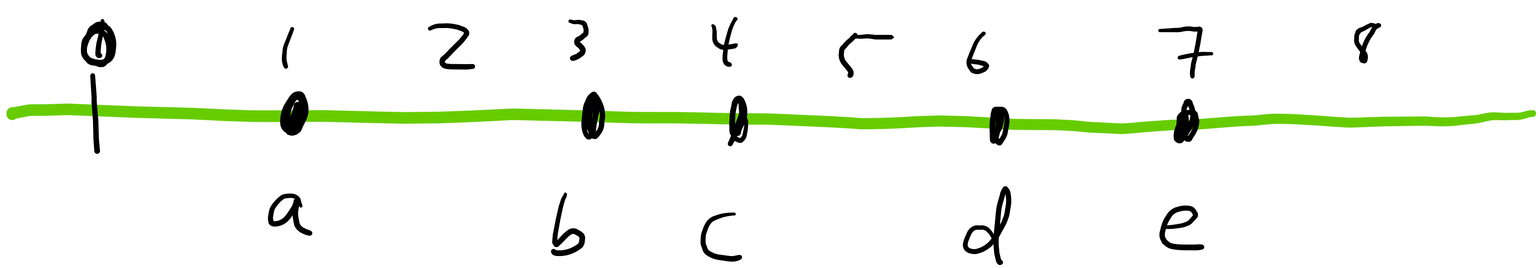

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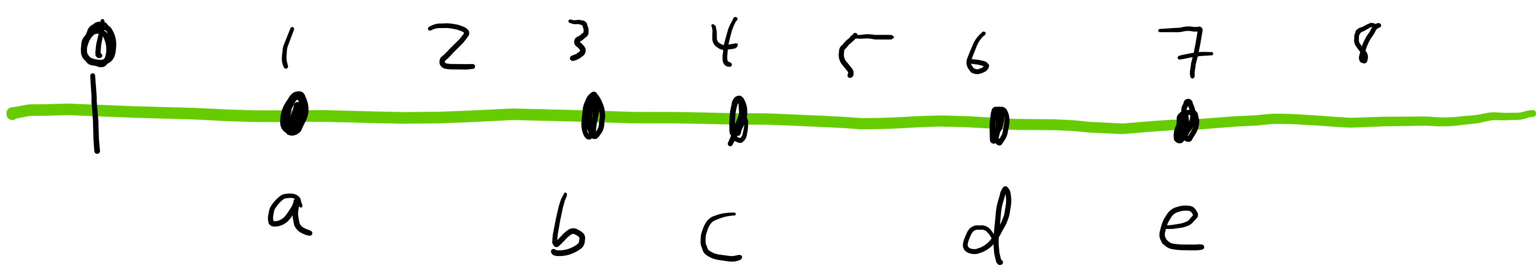
- $G$ is a *star-PCG* if it admits a $[d_1, d_2]$-PCG tree that is a subdivision of a star for some $d_1, d_2$.
- A graph G is a *double-threshold* graph if there exist $f : V(G) \to \mathbb{R}$ and $d_1, d_2 \in \mathbb{R}$ such that $uv \in E(G) \Leftrightarrow f(u) + f(v) \in [d_1, d_2]$.
- star-PCGs are exactly the double-threshold graphs.



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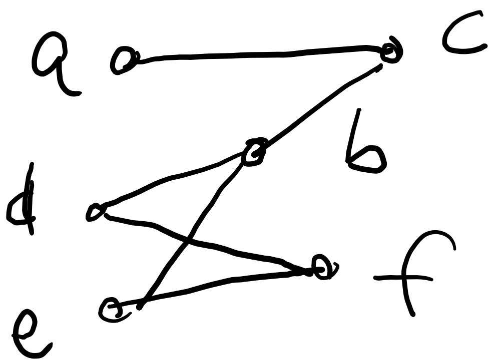
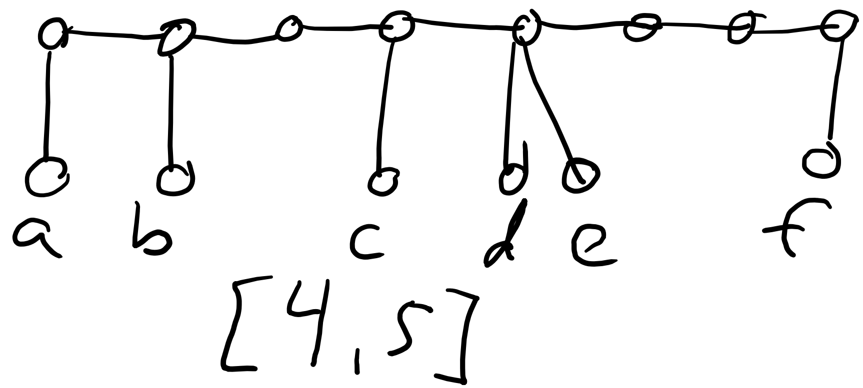
- star-PCGs are exactly the double-threshold graphs.
- double-threshold graphs can be recognized in linear time. [Kobayashi et al, 2022]
- How? Idea = double-threshold graphs form a subclass of permutation graphs, whose properties can be used for recognition.

# What if we restrict $T$?

- $G$ is a *caterpillar-PCG* if it admits a $[d_1, d_2]$-PCG tree that is a caterpillar for some $d_1, d_2$.



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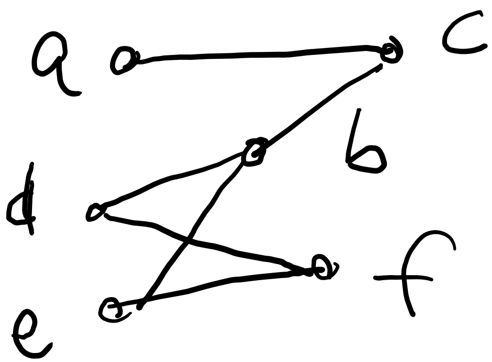


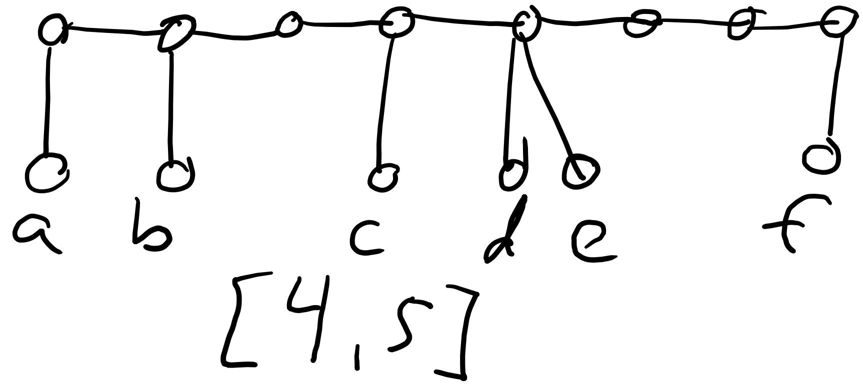


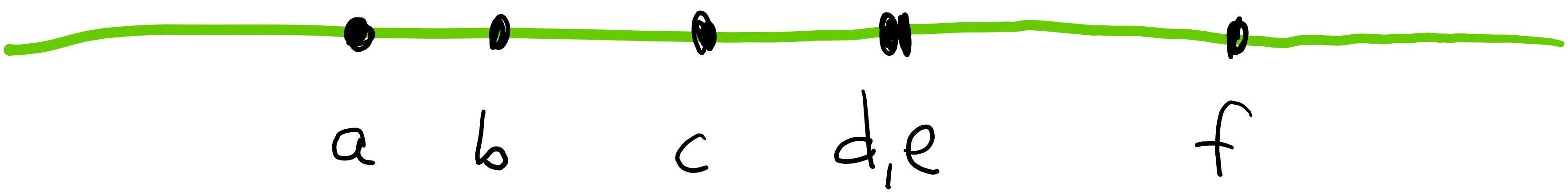

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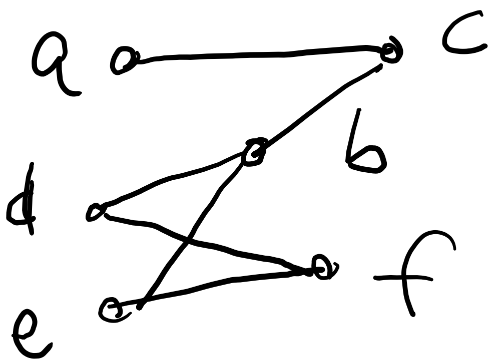


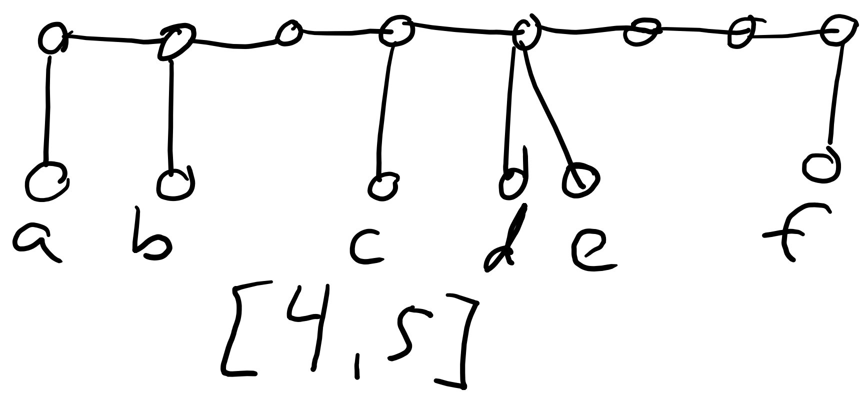


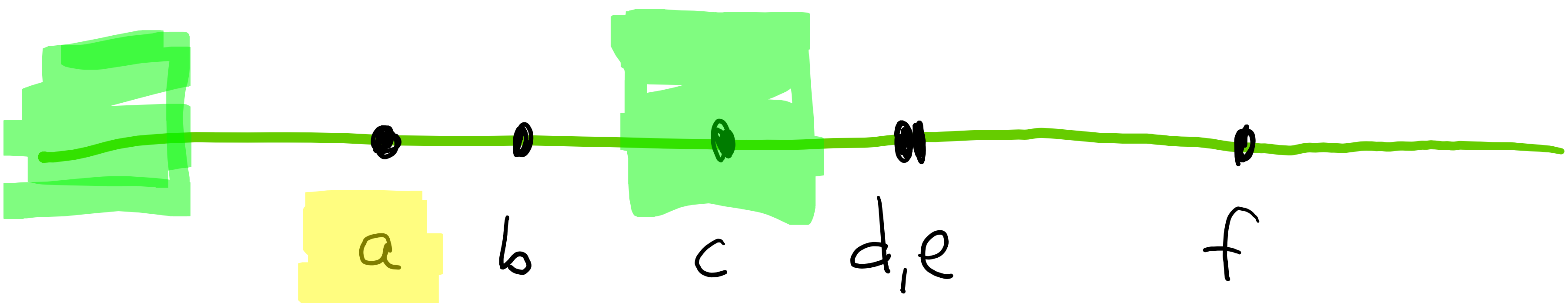

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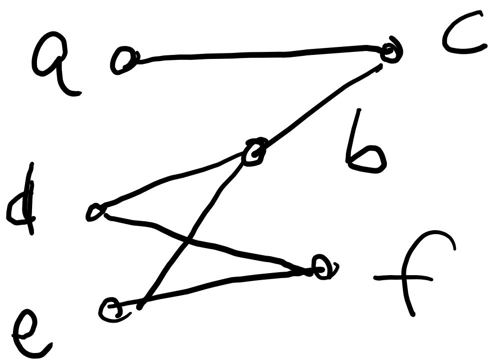


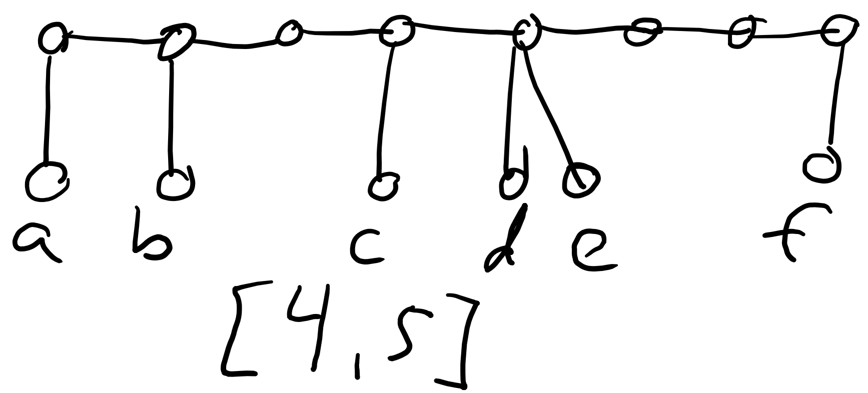


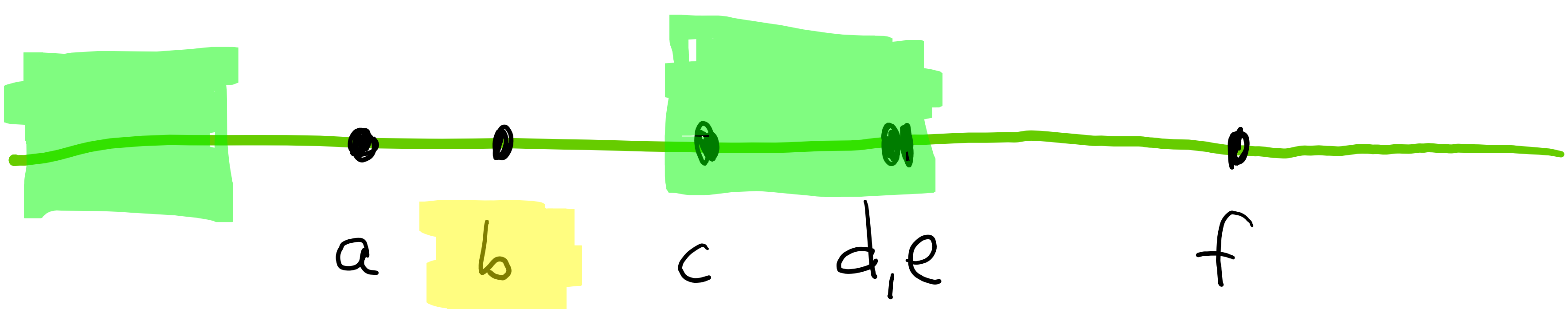

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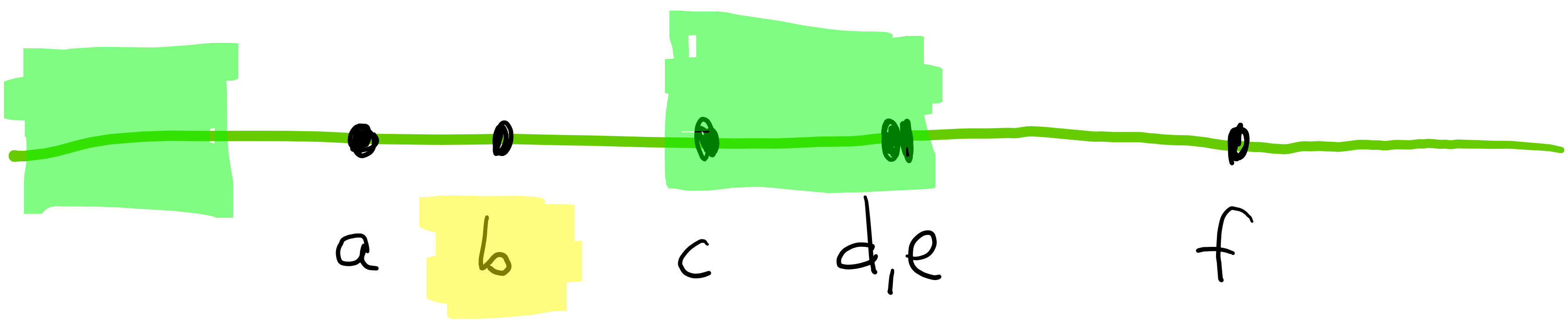
- $G$ is a *caterpillar-PCG* if it admits a $[d_1, d_2]$-PCG tree that is a caterpillar for some $d_1, d_2$.
- A graph G is a *double-difference* graph if there exist $f : V(G) \to \mathbb{R}$ and $d_1, d_2 \in \mathbb{R}$ such that
  $uv \in E(G) \Leftrightarrow |f(u) - f(v)| \in [d_1, d_2]$.
  (I called those double distance-threshold on Monday)
- caterpillar-PCGs are exactly the double-difference graphs.



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- **Open problem**: characterize double-difference graphs.